# a framework for analysing and comparing the mathematical engagement afforded in lessons

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This paper reports on the development and use of an analytical instrument which identifies mathematical affordances in the public tasks, questions and prompts of mathematics classrooms. I have explored the use of several frameworks which identify learning outcomes, structures of knowledge, mental actions, teaching actions and intentions and found that none of them give me access to the detail of what makes one maths lesson different to another for learners. From the experience of using these I devised a new analytical tool which unfolds patterns of participation afforded in mathematics lessons. This tool has been tested on several videos of lessons, and has been used by pre-service teaching students to analyse their own lessons.

## Development

This paper is a contribution to ongoing work in which I have come to view teaching as the creation, with learners, of micro-cultures in which mathematical activity is afforded and constrained, and to view learning partly as the shifts, changes and developments which take place in and through participation in those activities . In Watson (2004) I wrote about how mathematical micro-cultures can be described in terms of the activities they afford, the anticipations which might be structured by enabling constraints, and the attunement of learners towards patterns of participation - an ecological environment in which the presence of explicit variation and invariance in mathematical objects and signs contributes to the structure of activity. I showed that attention to the mathematical affordances created by teachers can be a powerful method for analysing how some teaching is more effective than some other teaching, and how learning can be understood by examining ways in which learners might participate in what is available in the learning environment.

Analysis of classroom incidents using these ideas allows the discipline of mathematics to contribute to the analysis in a powerful way, alongside socio-cultural perspectives. Over time, repeated patterns of affordance and constraint lead to the development of patterns of participation, so learners who are often given tasks which require exemplification and experimentation are more able to work in these ways as the norm than those for whom these expectations are rare.

It is with these understandings, which depend heavily on a view of mathematics as a disciplined collection of ways of thinking, participating and being (Freudenthal, 1991), that I approached the task of analysing a set of videos. Is there a perspective which is independent of subject content, teaching style, lesson structure or classroom culture which can be used to compare the nature of mathematical participation in lessons?

## Background

In a current three-year project with Els De Geest, Changes in Mathematics Teaching (CMTP), the target students are those who enter secondary school below national target levels of achievement but the central unit of analysis has shifted from individual teachers to mathematics departments. We are chronicling the stories of three teams of teachers who deliberately set out, in September 2005, to redeem a significant number of such students. Eventually we are going to describe their practices, and identify factors which appear to contribute to, or hinder, success. We have interviews with teachers and teaching assistants; interview and test data (where available) from some students; documents; observation notes of department meetings; copies of resources; videos of lessons; and so on.

For each of these data-types we first do a content analysis, followed by categorising the content using a range of perspectives. Much of the data from teachers is codified using third generation activity theory (Engestrom, 1998), seeing interaction between departmental and classroom activity as a site for identifying parameters of change. Activity theory allows us to ‘lay out’ the stories of the departments, and identify common and particular tensions between teachers, between schools and for each teacher and department over time, but this approach backgrounds important details. For example, we know from earlier projects that readers will ask ‘but what did they actually do?’

## Lesson videos

The analysis of the videos from the first year of the project is the focus of this paper. The purpose of videoing lessons was to collect a sample of classroom practices over the duration of the project to get some sense of the range and of any similarities and differences, or patterns, between and within schools. It is important to note that the departments appear to espouse similar overarching interests in the development of mathematical thinking. None of them chose ‘drill and skill’ as an approach to rescuing learners. How were we to analyse the videos to produce a full description of the range of practices in classrooms, at a level of detail which is informative for the research schools and more widely, especially as observing individual lessons gives little insight into how learners gradually become enculturated, over time, into the practices of a particular classroom?

My role in the project included analysis of the video, but I did not have an analytical tool to hand that I thought would be effective. The first stage of analysis of videos was straightforward, which was to produce an account of what I could hear and see which related to the unfolding mathematical story of the lesson. In other words, what utterances, actions and interactions between the teacher and others were publicly available to structure the mathematical activity? While making these accounts I had to work quickly and openly so that I could send them quickly back to the teachers to indicate the nature of the interest researchers were going to take in their teaching. Just as third-generation activity laid out the parameters of the systems within which teachers were working, so these analytical accounts laid out the public discourse of each lesson but did little else.

## frameworks

To situate my work in the literature I looked in a variety of places for suitable frameworks to inform the next stage of analysis. There are five main kinds of focus for these:

* learning outcomes,
* structures of knowledge,
* mental actions students might undertake,
* teaching intentions,
* teachers’ actions

Existing analytical frames can shape what we look at, but may mask detail, yet by using several frames successively, and reflecting on how they foreground and background aspects of the data, I become more articulate about the fine grain of commonalities, differences, and relationships between teachers, tasks and classroom practices and, more importantly, developed research questions.

Teachers expect that what they say, and the tasks they set, will help learners achieve certain learning objectives. It seemed sensible to start with Bloom *et al’*s (1984) taxonomy as this is currently ‘around’ in schools which are focusing on ‘learning’ as a whole-school issue.

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| **Bloom et al.’s taxonomy of learning objectives** |
| Knowledge Comprehension Application Analysis Synthesis Evaluation |

When applied to mathematics, Bloom’s taxonomy seems very crude. For example, it does not provide for post-synthetic mathematical actions, such as abstraction and objectification, although it could be argued that the reflection involved in evaluation could include those. However, in classrooms it is more likely that ‘evaluation’ derived from this model would be an affective and/or target-accounting process rather than a reflection on emergent learning which might encapsulate recent experience as a new mathematical conceptual entity. Blooms’ taxonomy also underplays knowledge and comprehension in mathematics, both of which are multi-layered and require successive experiences in different mathematical contexts. ‘Comprehension’ can mean anything from ‘understands how to do it’ to ‘understands its place in some overarching unifying theory’. ‘Knowledge’ can refer to results, techniques, concepts or behaviours. For example, what does it mean to have knowledge of equations? Knowing what an equation is, knowing how to work out what it represents, and knowing how to solve it are very different kinds of knowledge.

Another possible contender arises in the SOLO (Structure of Observed Learning Outcomes) taxonomy from Biggs and Collis (1982):

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| **Biggs and Collis’ SOLO taxonomy** |
| Pre-structuralUnistructuralMultistructuralRelationalExtended abstract |

This offers plausible links between what teachers offer and what learners might perceive: if learners are only offered unistructural situations (simple and obvious relations) they are less likely to develop multistructural understandings. This approach is more promising than Bloom’s for my aim to analyse micro-differences in teaching, in that it enumerates input and output variables, it prioritises relationships, and it allows for abstraction. These possible learning outcomes can be used to devise questions which make finer distinctions than the vague notions of ‘lower order’ and ‘higher order’ which are often found in the literature on questioning. Translated from a model of learning to a model of teaching, however, this taxonomy does not allow for the interplay between simple and complex examples, between symbols and images, and between examples and generalisation, which characterise mathematical activity. It is also true of any such taxonomies that it matters whose view you are taking. This central problem affects any attempt to use ‘learning outcome’ taxonomies to categorise teaching, and yet without complex articulation of learning, teachers cannot sensibly create or select tasks.

Several frameworks describe structures of mathematical knowledge. I will not rehearse them here, but in general they describe initial activity with mathematical objects and tools, then subsequent generalisation and abstraction of ideas at a more formal level. These models, especially those which include interaction as a means to shift between perceptual, spontaneous, and formal, scientific, conceptualisations, link an epistemology of mathematics to a constructivist psychology of logical learning.

This is well-illustrated in the Van Hiele model of geometric understanding (Usiskin, 1982) which describes the human activity of working mathematically by characterising visualisation, analysis, informal deduction, formal deduction and becoming rigorous as levels at which you can structure tasks for students, as well as levels of understanding.

Another non-hierarchical description of learning activity is developed in the Pirie-Kieren model of mathematical understanding (1994), which attempts to relate different kinds of mathematical engagement and allows for ‘folding-back’ to earlier levels rather than assuming a monotonic outward movement. However, their ‘layers’ cannot all be reasonably matched to teachers’ utterances.

The analytical frame derived for video analysis in the METE international project is more promising as a tool for focusing on mathematical prompts (Andrews, Hatch and Sayers, 2005). This gets close to the intentions of teaching through classifying features of mathematical meaning and structure without assuming that learners necessarily do what is intended. Thus it categorises what might be afforded and constrained in the public mathematical discourse. Their framework looks firstly at whether teachers emphasise and encourage:

* conceptual knowledge
* derivational knowledge
* structural knowledge
* procedural knowledge
* use of efficient methods
* problem-solving
* reasoning

Further, it focuses on pedagogic strategies which are exploited to work on these foci: activating prior knowledge, exercising, explaining, sharing, exploring, coaching, assessing and questioning.

This focuses on a finer grain of detail than the analytical frames used in the TIMSS seven-nation video study (Hiebert et al., 2003). In the latter study descriptions of typical national lesson types were constructed which enabled cross-national comparison. Further analysis which probed beneath superficial lesson characteristics found that lessons in the more successful countries (as measured by international tests) were characterised by high content level, coherence, structured argument and many opportunities for students to think, whatever the lesson format (Hiebert et al, 2003; Leung, 2006). These descriptions were simply too broad to be of use to us, but are a useful pointer towards the need to map mathematical development in lessons rather than only looking at behavioural, organisational and social norms. Closer to home, the Leverhulme project (Brown et al., 2001) found that analysis of observable lesson structures did not show strong correlation to the success of the mathematics teaching. They produced a detailed instrument for evaluating lessons which reflects the kinds of dimensions of good teaching which were reported in our earlier study (Watson, De Geest and Prestage, 2003). The Leverhulme instrument required certain value judgements and assumptions about ‘better’ teaching which were irrelevant to our purposes, but was useful to aid thinking about the qualities of mathematical activity. For example, if a teacher is encouraging links to be made between mathematical entities, what is the significance of the links and how is that being done? Most importantly, I found that, with the possible exception of the Leverhulme instrument (depending on interpretation), the sense of conceptual construction that is evident in models of mathematical understanding was not embedded methods for analysing teaching.

**Identifying the mathematical affordances of lessons**

The process of trying to use existing frames helped me to pose this question: what opportunities to act mathematically are afforded and constrained by the public tasks, questions and prompts in mathematics classrooms? In other words, to construct my analytical categories I start from mathematics rather than from teaching, or from learning outcomes. The analytical instrument which I devised and tested is a list of dimensions of mathematical orientation which takes as given that the aim of teaching is to enable learners to act mathematically.

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| **Make or elicit declarative/nominal/factual/technical statements*** Say what the lesson is about
* Information giving
* Define terms
* Know/ask facts, definitions, techniques
* ‘Research’ facts, definitions, techniques

 *shift: remember***Tell learners to*** Imitate
* Follow procedure
* Make
* Find answer using procedure
* Give answers

 *shift: fluency, report/record actions* **Teacher directs learner’s perception/ attention*** Tell/show objects (including worked examples) which are perceived as having a single feature
* Tell/show objects which are perceived as having multiple features
* Tell/show multiple objects
* Indicate identification of characteristics/properties
 | * Indicate classification
* Indicate comparison
* Indicate identification of variables and variation
* Summarise what has been done

*shift: public orientation towards concept, method and properties***Ask for learner response*** Use prior knowledge
* Find answer without knowing procedure
* Visualise
* Seek pattern
* Compare, classify
* Describe
* Explore variation
* Informal induction
* Informal deduction
* Creating objects/examples with one feature
* Creating objects/examples with multiple features
* Exemplification
* Express in ‘own words’
* Say what to think about

*shift: personal orientation towards concept, method or properties* | **Discuss implications*** Varying the variables deliberately
* Adapting procedures
* Identifying relationships
* Explication/ Justification
* Induction/ Prediction
* Deduction

*shift: analysis, focus on outcomes and relationships***Integrate and connect*** Clarify
* Associate ideas
* Generalisation
* Redescribe
* Abstraction
* Objectification
* Formalisation
* New definition
* Summarise development of ideas

 *shift: synthesis, connection***Affirm*** Adapt/ transform ideas
* Application to more complex maths
* Application to other contexts
* Prove
* Evaluation of process

 *shift: rigour, objectification, application* |

The bold headings are dimensions of mathematical pedagogic orientation, and classify the kinds of mathematical focus I identified in videos, organised with reference to hierarchical models of mathematical structure. The words in normal text classify public tasks, questions and prompts within each overarching orientation. These were derived from watching the videos, informed by earlier work on mathematical activity (Watson and Mason, 1998), and by incorporating aspects of the models of mathematical knowledge outlined above. I see these as the range of possibilities within each dimension. In particular, I tried to include the relationship between being showing something to learners, indicating aspects of it, learners attending to these, and finding aspects for themselves. The italicised words are a summary of the kinds of shift a learner might be hoped to make during such activity, but I have avoided assumptions about what students actually do learn.

Although the list is hierarchical in terms of progress towards mathematical application and/or abstraction, it is intended to be complex rather than linear. It is not a model, as it does not have the essential connecting and relating features of a model. Rather it is the contents for a future model. Teaching and learning is not assumed to be unidirectional within it, nor should it be. For example, the stage which some might call ‘vertical mathematisation’, or ‘objectification’, is written before application which, for some, is seen as less arduous mentally. This is because in school schemes of work the final experience of a ‘new’ mathematical idea is not its objectification or encapsulation, but its re-appearance later in the curriculum, or in another subject, as a tool.

I shall now apply the instrument to a lesson:

In one of the videos from our own CMT project, a lesson started with the class being asked what they thought of when they saw the word ‘algebra’. Much of the lesson then evolved from the nature of their contributions. In my analysis the sequence of kinds of activity afforded is:

Association of ideas, prior knowledge, exemplification, comparison, identifying relationships, new definitions, defining terms, copying, doing numerical examples, informal induction, formalising, creating objects with one feature, being shown objects with multiple features, classifying, explication, applying to other contexts.

This lesson ranged across all parts of the list, the numerical examples being offered in the middle of the lesson as a key component of the development of new ideas, rather than as a precursor to, or a manifestation of, them. What kinds of learning shifts did the lesson afford? It started with an interplay between personal and public orientations until sufficient evidence was derived from comparing examples to synthesise a new (to them) definition of an aspect of algebra; then, after recording numerical examples, analysing and synthesising new ideas. These new ideas were subjected to a similar interplay of personal and public orientation with, finally, some examples of application in other contexts.

The complex interplay and the emergent nature of the teaching-learning interaction demonstrated in this lesson fits with what Vygotsky was aiming at with his notion of ZPD (Valsiner, 1988, p. 144). This lesson demonstrates substantial use of learner exemplification to provide starting points for the development of new conceptual ideas (Watson and Mason, 2005). There was explicit progress from what is already known to new ideas, culminating in application of a ‘new’ idea to a ‘new’ context.

Development of the analytical tool is work-in-progress, including the way the ideas are expressed within it. In the analysis of the first year’s videos we find that it enables us to be specifically mathematical in our descriptions of classroom interaction, and to map patterns of opportunity for mathematical activity in lessons. More than that, we find it also supports a developing picture of the practice of being a mathematics student in school, as different from ‘being a student’ and ‘being a mathematician’. Being a mathematics student involves becoming attuned to the kinds of participation afforded in these lessons.

To test the instrument further, pre-service mathematics teachers were asked to use it to ‘map’ the nature of mathematics in one of their recent lessons. They drew dots next to events in their lessons and joined them up with a directed path. Then they justified their choice of path to their neighbours – for example, what are the reasons for starting with some of the activities described on the left, or on the right? Teachers who found that they only used the earlier (leftmost) parts of the list could see easily that they had missed opportunities to prompt other kinds of mathematical thinking.

This kind of analysis enables us now to compare very different lessons in terms of the kinds of mathematical participation afforded, the kinds of shifts learners might make in the nature of their participation, and the ways in which these are developed during a lesson, through the affordances and constraints of the mathematical tasks, questions and prompts. Furthermore, the importance of the order of different features of lessons, and their placing in the sequence, can be conjectured through identifying how these influence what is afforded. The process of devising and using the instrument has itself led to new insights about how a mathematics lesson can be viewed.

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1. Other versions of this paper have been published in BSRLM proceedings, and submitted to RME [↑](#footnote-ref-1)