Student-generated examples in the learning of mathematics

ARTICLE *in* CANADIAN JOURNAL OF SCIENCE MATHEMATICS AND TECHNOLOGY EDUCATION · JANUARY 2002

DOI: 10.1080/14926150209556516

2 AUTHORS:


Anne Watson
University of Oxford


John Mason
University of Oxford

Mason, J. & Watson, A. 2001, Stimulating Students to Construct Boundary Examples, *Quaestiones Mathematicae,* 24 Suppl 1, p123-132.

Stimulating Students To Construct Boundary Examples

John Mason
Open University, UK
j.h.mason@open.ac.uk

Anne Watson
Oxford University, UK
anne.watson@edstud.ox.ac.uk

Mathematics Subject Classification (1991):00A35, 97C90, 97D40

Abstract: We address three common difficulties encountered by students: not appreciating the necessity of conditions in a theorem before using it; using non-generic examples as if they were generic; and ignoring counter-examples as pathologies. We propose the conjecture that students would be more likely to remember and to appreciate the importance of conditions, if they were stimulated to construct examples for themselves which show why each of the conditions is necessary. Furthermore, constructing their own examples is likely to prompt them to explore the space of possibilities admitted by definitions, and hence to appreciate both the range of situations encompassed by the definition, and the force of both definition and theorem. We illustrate our conjectures with some particular but generic tasks for students, and we use these to consider what is involved in constructing examples, leading to ways to support students in learning how to construct them for themselves.

1. Method Our method of enquiry is to identify phenomena we wish to study, and to seek examples within our own experience. We then construct task-exercises to offer to others to see if they recognise what we find ourselves noticing. Through refinement and adjustment of task-exercises in the light of experience and of reading relevant literature, we both extend our own awarenesses, and offer others experiences which may highlight or even awaken sensitivities and awarenesses for them. These sensitivities and awarenesses may inform their future practice. As task-exercises are developed and shared, actions which exploit what is noticed become part of regular teaching for the benefit of students. Our method does not attempt to capture or cover the experience of readers. Rather it aims to make contact with that experience, perhaps challenging interpretations, perhaps pointing to features not previously noticed. The data of this method are the experiences generated, the sensitivities to notice which are enhanced. If you recognise at least something of what we are talking about as a result of having worked on these problems, you may be stimulated to look out for similar experiences in the future, and over time, begin to act upon what you notice. Validity in this method lies in you finding your actions being informed in the future, not in what we say.

The task-exercises which follow are intended to bring to the surface various features about the construction of examples to meet constraints, starting from the premise that if you simply ask students out of the blue to construct a mathematical object, they are likely to find it very difficult if not impossible. This may lead a tutor to lose confidence and to conclude that 'students can't do this sort of task'. The effect is a move from impoverished past experience (students are not aware of the construction of objects) to a continuation of impoverished experience (students are not called upon to construct examples, because 'it is too hard'), and so the cycle continues. The aim of this paper is to locate ways of breaking out of this cycle.

2. Task-Exercises

1 A Routine Problem: solve the differential equation $f''(x) + b f'(x) + c f(x) = 0$.

2 *A Mean Problem*: Observe that $\int_0^2 (1-x)\,dx = 0$. Generalise!

3 *A Divisory Problem*: Find all the positive integers which have an odd number of divisors.

4 *Another Square Problem*: Given two distinct straight lines L_1 and L_2 and a point E not on them, construct a square with one vertex at E, and one vertex on each of L_1 and L_2.

5 *Rolle Points*: Rolle's theorem tells us that any function differentiable on an interval has a point in the interior of that interval at which the slope of the function is the same as the slope of the chord between the points on the curve at the ends of that interval. Where on that interval would you expect to look for such a point? For example, are there any functions for which the Rolle point of every interval is the midpoint? A natural question to ask is whether there are any functions for which the Rolle point on any interval is, say, 2/3 of the way along the interval, or more generally ρ of the way along.

6 *Inflection points*: A common method for finding inflection points of a curve which is at least twice differentiable, is to differentiate twice and set equal to zero to find the abscissa. Sometimes this gives a correct answer for a correct reason, sometimes it gives a correct answer for a wrong reason, and sometimes it gives an incorrect answer. Construct examples which exemplify these three situations, and also a family of examples which include all three in each member, and thus might bring students up against these different possibilities. Must a function be twice differentiable to have an inflection point? What about members of the family $x^k\sin(1/x^2)$?

3. Specific Comments The first task (*A Routine Problem*) succumbs to a standard procedure, and indeed, few students are likely to solve it without having been shown a technique (both in particular and in general). But in being shown the technique, few probably ever think about what it is doing in terms of displaying a strategy: when in doubt try something you are familiar with and be prepared to adjust it, or better, express in symbols a general class of objects and then see if the constraints on parameters can be resolved. They see it simply as the solution to a problem, not a process of construction. Yet what it achieves is the construction of not just a single function but of a whole class of functions. The theorems of the topic show that this class includes all solutions. In other words, their structure characterises all solutions to the constraint.

For the second task (*A Mean Problem*), some students will think of varying the function but not the limits, while others will vary the limits but not the function. Some will try different specific linear or other functions, while others will write down a general (linear) function and then see what the integral condition imposes on the parameters. Some will remain with polynomials (a very fruitful as well as manageable constraint), while others will extend to familiar periodic functions. What they choose to vary and what they leave unvaried suggests where they are confident and where they are not. In any case, varying two things at once requires considerably more confidence and attention than varying just one.

Shifting attention to the fact that for a linear function, *any* interval with the zero point as the mid point will meet the constraint may raise an auxiliary question of whether there are any functions for which, for any interval with the zero point say 2/3 of the way along that interval, the integral will be zero, thus producing a sort of lopsided symmetry.

This problem nicely illustrates the situation in which incremental alterations can be made in the problem on the way to considerable generality. If the tutor's guidance keeps pushing the student to more and more variation, students will find themselves becoming aware of a wider and wider class of functions from which to choose when looking for examples meeting other constraints in the future. For example, if the tutor announces the existence of a function (held in a sealed envelope) known to be specified on [0, 1/2] and integrable there, can the student augment it to one on [0, 1] with the 0 integral property. This exposes students to a construction strategy for building functions by gluing them together at specified points.

The third task (*A Divisory Task*) is intended to highlight differences between leaping to the general but getting bogged down, and specialising to some simple examples to get a sense of what might be going on. For example, you can write down the general shape of a number (in a form which enables you to count factors) as the product of primes to various powers. You can then calculate the number of divisors, impose the condition that it must be odd, and look at what this says about your number. Alternatively you can locate some numbers which meet the conditions, and see what they have in common. Once you detect a pattern you can make a conjecture, and then try to justify that conjecture, perhaps using further work on examples as a source of insight. A related approach involves finding a way to count the number of divisors of a number from a list of its factors. In the process of investigation, students are likely to try various numbers, and perhaps even to begin seeing that writing down a number in base ten is not always the best way of exposing structure, enabling them in the future to construct numbers meeting constraints by choosing between different representations.

The fourth task (*Another Square Problem*) can be approached using a device which works extremely well in the context of dynamic geometry software. If a problem is too hard, remove a constraint, and then look for a locus. Here we require remove the requirement that one vertex lie on L_2, and we examine the loci of the other vertices of all squares meeting the remaining constraints. The fact that the loci are straight-lines informs us about what we need to prove, and also points the way to building a construction (Love 1996) The same strategy works in algebraic contexts. You remove or weaken a constraint, usually by expressing the more general using a parameter, and then look at the class of solutions this produces to see if you can meet the original constraint by suitable choice of parameter.

For the fifth task (*A Rolle Problem*), finding why there are no polynomials of degree larger than 1 for which ρ can be anything other than 1/2 is instructive because of the need to make use of the 'over every interval' condition. This is likely to call upon using the technique for forcing a polynomial to be constantly zero. Again one method is to try some simple functions (clearly a linear function works for any ρ, but might there be any others?), eventually stumbling on quadratics for the midpoint but failing with cubics and higher. Moving to exponential functions $e^{\lambda x}$ also fails, but by weakening the constraint, solutions to an adjusted problem can be found. For example, it *is* possible for intervals of a fixed width h to have a constant ρ (depending on h), for which every interval of width h has a Rolle Poin*t* ρ of the way along the interval. The possible values for ρ turn out to depend on λ, so that for any specified $\rho > 0$ there is a class of exponential functions which have the Rolle Point of the way along the interval for every interval of specified width (depending on ρ). This approach is analogous to the geometrical strategy in the previous problem: You weaken a constraint and try to solve that problem.

This approach to mathematical exploration is classic mathematical behaviour, but not one which is often drawn to students' attention. Consequently they are hampered if and when they are asked to construct an object meeting specified constraints for themselves. The method of trying a simple case (a pure quadratic, a pure linear), and then when one works generalising to a broader class (all quadratics) and when one fails, seeing why none of its class can work, is quintessential mathematical thinking.

Another approach exploits Taylor's theorem, but leads to the question of whether there are any solutions which do not have a Taylor approximation.

In the sixth task (*Inflection*), students are invited to exemplify the incorrect use of (part of) a technique, and to 'bury' a wrinkle or potential difficulty which is the subject of conditions surrounding the statement of the technique. Students whose method of finding inflection points of a curve is to differentiate twice and set equal to zero to find the abscissa get a correct answer for a correct reason on functions like $f(x) = x^3 - x$ where the slope at the inflection is *not* zero, though many expect an inflection point to have a zero slope. Students using the same method on $f(x) = x^3$ will get a correct answer for an incorrect reason (you have to reason about the change in $f'(x)$), and on $f(x) = x^4$ they will of course get an incorrect answer. The class of functions $x^k e^{(-1/x^2)}$ is not one many students are likely to come up with, but running into these functions in various contexts involving construction may alert them to their use (Michener 1978).

Trying to bury all three possibilities in one function could lead students to appreciate the underlying structure of what is going on (if the second derivative has repeated roots, the parity of the repetition determines whether there is an inflection or not). Such an example could then be used on other students to see if they are caught up in the mechanics of the technique and do not notice the wrinkles, or whether they correctly apply the technique and its conditions. Students might also wish to exemplify inappropriate use of the technique due to the function failing to have a first or perhaps a second derivative!

Caunt (1914 p137) gives the conditions of a point of inflection as $f''=0$ and changes sign in passing through it, or $f''=0$ and $f'''\neq 0$. He then draws up a table showing the effects of combinations of f' and f'' being positive, negative or zero, with illustrations. He also points out that an inflection point is a "point at which the tangent passes through three 'consecutive points' on the curve". SMP (1967 p212) distinguishes between stationary points which are also points of inflection, and oblique points of inflection, and suggests constructing a table showing the sign of the gradient as x increases. Neil & Shuard (1982 p54) first give an intrinsic definition (" a point at which the graph changes the direction in which it is bending"). They then give an intrinsic definition (f'' changes sign), and two warnings: that students often assume that the slope has to be 0 at an inflection point, and that "consideration of the second derivative should not be over emphasised".

4. General Comments Any technique which yields answers illustrates ways to construct mathematical objects. Solving a first order linear differential equation, finding an integral, disentangling eigenvalues and eigenvectors for a matrix, and changing basis are all examples of constructing an object with specified properties. Thus constructing mathematical objects is something students have seen and done a great deal of, yet for the most part they are unaware of it in these terms.

Curriculum topics can be seen as domains in which people have worked out techniques for resolving problems, usually through constructing objects meeting certain constraints. Where those techniques are deemed sufficiently simple, they are taught to students. But the teaching removes the creative aspects of problem resolution in favour of mechanical manipulations, and this means that students do not have to think about the class of objects from which they are trying to select a particular one. Since students are not used to thinking this way, they are often at sea when asked to construct an object for themselves having certain properties.

It follows that if students are going to be expected to construct other examples for themselves, they need to be aware of techniques as tools to call upon for this purpose. But there is a second level awareness which could also be offered to students. Where do techniques for solving specific classes of problems come from? How are they found?

Behind many of these techniques lies a very ancient and powerful principle, derived from the ancient roots of algebra: you describe what a general object looks like, then you impose constraints and use these to locate values of parameters. In arithmetic one proceeds from the known to the unknown through making calculations (synthesis); in algebra one proceeds from the unknown to the known (analysis): starting from the as-yet-unknown, expressing calculations as if one were checking a proposed solution, revealing relations and equations which are then resolved through algebraic techniques. Theon of Alexandria may have been the first to distinguish between *analysis* (beginning work with "the assumption of what is sought as though it were granted …" quoted in Klein 1932, p155) and *synthesis* (beginning work with "the assumption of what is granted …" quoted in Klein 1932, p155), but the idea has been taken up many times since. For example by Mary Boole (Tahta 1972 p55) who spoke of "acknowledging ignorance" by denoting what is not known by a label and then manipulating that label.

5. Constructing Boundary Examples Constructing boundary examples is a special case of constructing mathematical objects, which is why the bulk of the paper has focused on the more general problem. The problems discussed revealed among other things that the strategy of expressing a general object of particular form and then seeking constraints on the parameters in order to locate an object which meets the given constraints is powerful and pervasive. You start with maximum freedom (generality) and then you impose constraints. Another form of it is weakening a constraint to reveal a class of solutions, which sometimes enables a solution to be found to weaker constraints while the tighter constraint remains unsolved. This is what mathematicians consider to be 'progress'.

When students are offered examples to illustrate theorems, and even where these are boundary examples because they show why constraints are required, students have to make sense of what the examples are exemplifying. For example, MacHale (1980) pointed out that many students dismiss $f(x) = |x|$ as a pathology when shown that it is not differentiable at one point. They continue to act as if most functions are actually differentiable everywhere, and reasonably so because the particular example seems to them contrived. Whereas to the tutor it is patently obvious what is being exemplified and why the example is a boundary example (the same construction could be used to produce a wide range of functions differentiable at all sorts of different points), students do not usually have the same sense of generality as the tutor, and so may not appreciate the examples in the same way, at least without some help (eg. functions glued together continuously but not differentiably) at a range of points. In order to appreciate the particular, the students need to appreciate the general which it particularises, yet it is through the particular that they begin to appreciate the general!

If students are challenged to make us of the idea behind $f(x) = |x|$ to construct functions which are differentiable everywhere except at 1, 2, 3, … points, or at all points in $\{1/n: n \text{ an integer} \neq 0\}$, they may begin to appreciate the immense number and range of examples signified by and constructible from the one idea.

Furthermore, when an example is given, the students may alight on specific features which the tutor knows are irrelevant. Mariotti (1992) and Fischbein (1993) have studied this in detail in the context of geometry. But the same principle applies to the algebraic. Watson & Mason (1998) developed a technique which is specifically designed to alert students to the possibility that they have a narrow range of examples to call upon, and that they may be using overly special cases as their exemplars of properties. For example, here is a task-exercise which often serves this purpose with respect to continuous functions.

- Sketch the graph of a function on the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1];
- Sketch the graph of a differentiable function on the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with one of its extremal values at the left end of the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with both its extremal values at the end points of the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with its extremal values at the end points, and with a local maximum in the interior of the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with its extremal values at the end points, and with a local maximum and a local minimum in the interior of the interval [0, 1].

Now comes the interesting part! Work your way back through the examples, making sure that at each stage your example does *not* satisfy the constraints which follow! Thus your first example must be a function but must not be continuous; your last but one example must have a local maximum in the interior but not a local minimum. Finding that a set of constraints seem mutually incompatible is an excellent way to generate a conjecture leading to a little theorem. The structure of this kind of task forces students to become aware of a more general class of examples than they may have considered the first time.

Another device for developing awareness of the general class of objects which meet a given constraint is to ask students to construct a simple object, then a peculiar object (having something unusual about it; perhaps some feature that no-one else in the class is likely to think of), then to try to describe a general object of the class (Bills 1996). For example,

- Write down a number leaving a remainder of 1 on dividing by 7;
- Write down a number leaving a remainder of 1 on dividing by 7, but which you think no-one else (who is present, who is alive, …) will think of;
- Write down a description of all numbers leaving a remainder of 1 on dividing by 7.

Now we could go on and multiply two of these together and discover that it is of the same form, perhaps even exploring the notion of a prime in this restricted set of numbers closed under multiplication.

Another example related to *inflection* could be:

- Write down a cubic which passes through the origin and has an inflection point there.
- Write down a cubic which passes through the origin and has an inflection point there but which no-one else in the room is likely to write down.
- Write down all cubics which pass through the origin and have an inflection point there.

There is an opportunity to explore how much freedom there is in specifying the abscissae of inflection points for polynomials together with the slopes at those points.

In both these tasks, the effect of the second construction is to prompt students to think not just of one but of several, and then to become playful with some method of generating them. In the process they come into contact with the general.

A context for constructing boundary examples can also be provided by choosing several definitions, or candidates for definitions, of a single concept, and asking students to construct examples which distinguish between the definitions where possible. For example with inflection points, the definitions mentioned so far are but five among many others. A task can be constructed by listing the definitions and asking for examples which distinguish between the various definitions:

A point at which the graph changes the direction in which it is bending.

The slope of the tangent changes from increasing to decreasing or vice versa at the point.

A point at which the tangent passes through three 'consecutive points' on the curve.

A point at which the second derivative changes sign.

Either the second derivative is zero and changes sign in passing through it, or else $f'' = 0$ but $f''' \neq 0$.

Task: Construct examples where possible to show in what ways these definitions differ from each other, and examples which illustrate possible inflection points which are not covered by each definition. In what contexts might each be a sensible definition to choose?

6. Summary Using task-exercises selected to reveal or highlight aspects of example construction, we have suggested that any technique for solving a class of problems can be seen as a construction process, and that adopting this perspective would alert students to this way of thinking more generally. The mathematical strategy of writing down a general object and then imposing constraints to see if parameters can be chosen to meet those constraints mirrors the shift from synthesis to analysis identified by Theon in early Greek geometry, and lying at the heart of the development of algebra as a powerful problem solving device. Offering a particular example and then guiding students to use that idea to construct families of objects with similar properties encourages students to experience the significance of the 'single example'. If tutors become aware of construction techniques such as using glued functions to piece together a function with several specified properties, and if they also become aware that this can be generalised (meet some constraints with part of an object, and other constraints with another part and stick them together somehow), they can then use that awareness to stimulate students into using it, then draw students' attention to the use. Three structures were exemplified for tasks which encourage students to extend their access to paradigmatic and generic examples. Constructing classes of examples leads naturally to the problem of classifying or characterising all possible such examples.

REFERENCES

L. BILLS, The use of examples in the teaching and learning of mathematics. In L. PUIG and A. GUTIÉRREZ (Eds.) *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education,* 1996 2.81-2.88, Valencia: Universitat de València.

G. CAUNT, *An Introduction To The Infinitesimal Calculus With Applications to Mechanics and Physics*, Oxford: Oxford University Press 1914.

E. FISCHBEIN, The Theory of Figural Concepts. *Educational Studies in Mathematics*, **24 (2)** 1993 139-162.

J. KLEIN, translated by E. BRANN (1992), *Greek Mathematical Thought and The Origin of Algebra*, New York: Dover 1934.

E. LOVE, Letting go: an approach to geometric problem solving, in L. PUIG & A. GUTIÉRREZ (Eds.) *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, 1996 3.281-3.288, Valencia: Universitat de València.

D. MACHALE, The Predictability of Counter-examples. *American Mathematical Monthly*, **87,** 1980 752.

M-A. MARIOTTI, The Dialectical process Between Figures and Definition in Social Interaction in the Classroom, *Second Italian-German Bilateral Symposium on Mathematics Education*, Orbeck, April 21-26 1992.

E. MICHENER, Understanding Understanding Mathematics, *Cognitive Science* **2** 1978 361-383.

H. NEIL, & H. SHUARD, *Teaching Calculus*, Glasgow: Blackie 1982.

SMP, *Advanced Mathematics*, Cambridge: Cambridge University Press 1967.

D. TAHTA, A Boolean Anthology: selected writings of Mary Boole on mathematics education, Derby: Association of Teachers of Mathematics, 1972.

A. WATSON, & J. MASON, *Questions and Prompts for Mathematical Thinking,* Derby: Association of Teachers of Mathematics 1998.