**Reasoning about variables in 11 to 18 Year Olds: Informal, Schooled and Formal Expression in Learning about Functions**

*This study examines expressions of reasoning by some higher achieving 11 to 18 year old English students responding to a survey consisting of function tasks developed in collaboration with their teachers. We report on 70 students, 10 from each of English years 7–13. Iterative and comparative analysis identified capabilities and difficulties of students and suggested conjectures concerning links between the affordances of the tasks, the curriculum, and students' responses. The paper focuses on five of the survey tasks and highlights connections between informal and formal expressions of reasoning about variables in learning. We introduce the notion of 'schooled' expressions of reasoning, neither formal nor informal, to emphasise the role of the formatting tools introduced in school that shape future understanding and reasoning.*

**Introduction**

Knowledge of functions has often been described as one of the big ideas of mathematical learning (e.g., Kuntze et al., 2011). In our project we are concerned with the development of functions knowledge and in particular with how that knowledge develops across the years of secondary/high school. Development depends on the nature of particular curricula and forms of pedagogy, though probably not entirely. We investigated development by working with classes across the age range in secondary schools. A survey was developed in collaboration with the teachers who taught in these classes in two countries, Israel and England, which have different curricula with respect to functions. In this paper we focus only on the data from England to develop conjectures about a particular phenomenon; reports that draw on the Israeli data particularly or comparatively can be found elsewhere (give two refs, one comparative and one only Israel).

The survey had to consist of questions that could be accessed by the youngest students but would also address and examine progression in concepts relevant for functions right through to the oldest students. In this paper we summarise progression in expressions of reasoning about variables across five tasks in two English schools. We describe some aspects of learning towards functions by relating students' responses to the presentation of tasks, and we make conjectures about the formality of understanding and expression. Finally we identify 'schooled' expressions of reasoning, neither formal nor informal, to emphasise the role of the formatting tools introduced in school that shape future understanding and reasoning.

**Literature Review and Theoretical Considerations**

Our theoretical review anticipates aspects of our analysis and discussion and focuses mainly on reasoning about variables and adolescent learning of the formal concepts associated with functions, by which we mean functions defined within conventional mathematical language and expressed in mathematical symbolism. We begin by outlining key ideas in the research on variables, including identifying variables, forming relations between variables ­- correspondence and covariation - and related aspects of rate of change and gradient, students’ capabilities, and difficulties. Throughout this paper we are limiting our discussion to functions that are commonly studied in school. We continue with a discussion about adolescent learning of formal concepts. We close our literature review by introducing constructs from Leinhardt et al. (1990) - used by them to discuss tasks in functions, graphs, and graphing - which were important in our study.

**Reasoning about variables**

One strand of research on students' understanding of variables is about modeling material phenomena by controlling physical or time variables, and/or using spreadsheets and graphing software in a modeling context. In the school level studies we have seen, the variables are usually simple and countable or measurable, either discrete or continuous, and there is a strong argument made that even young children can adopt a modeling perspective given a suitable physical experiment (e.g. Ainley & Pratt, 2005; Dreyfus & Eisenberg, 1983; Karplus, 1978). In such situations the relationship between informal knowledge arising from describing a situation (such as saying where and why 'it goes up' or 'it goes down', or sketching a likely graphical representation, or describing the physical effects of varying one parameter) and formal knowledge expressed mathematically is not clear (Dreyfus & Eisenberg, 1983). While there is a suggestion that situation modeling experiences make later understanding of formal functions easier, we have been unable to find evidence of that. While this body of research adequately describes children's progression towards symbolic representation of known physical variables, we have been unable to find research which describes students move towards understanding purely abstract variables, or how they progress from understanding variables as measurable quantities, such as temperature or mass, to variables such as price per unit, density and speed. We are calling the former 'simple' and the latter 'compound' because at least two quantities are involved[[1]](#footnote-1). In our survey, therefore, we included abstract and contextual, continuous and discrete, and simple and compound variables.

Once variables are understood, students need to understand how the dependent and independent variables relate to each other. There are two general approaches to creating and conceptualizing functional relationships (e.g., Confrey & Smith 1994, 1995; Slavit, 1997). One approach is through understanding *covariation*, the way the dependent and independent variables change together. Movement from *y*m to *y*m+l is coordinated with movement from *x*m to *x*m+l. In tables of values, it involves the coordination of the variation in two or more columns as one moves down (or up) the table; with a graph, it involves understanding changes in vertical value as one moves horizontally. This covariation approach can be contrasted to a *correspondence* approach of connecting particular  *x* and *y*-values and describing the connection as a rule. In our survey therefore we included tasks that could be approached using either a correspondence or covariational approach. Covariation is a precursor to calculus via understanding the derivative (Cottrill et al., 1996; Kaput, 1992; Thompson, 1994b; Zandieh, 2000) and also fits well with modeling natural phenomena, where data typically presents changes in a phenomenon (e.g., Carlson, Oehrtman, & Engelke, 2010; Thompson, 1994a). A full understanding would include connections between these approaches. We have described our view of these connections elsewhere:

we see ‘rate of change’ as an instantiation of a relationship in which changes in one variable can be expressed formally or numerically in terms of changes in another variable, where covariation more generally might not provide this precision. In situations where rate of change can be calculated, it might be correctly deduced through a procedural approach such as comparing step sizes or drawing ‘gradient’ triangles without any awareness of what it means in terms of covariation; alternatively, a student might express a situation qualitatively in terms of covariation but be unable to operationalise this idea as a rate of change (Ayalon et al. IJSME paper and page number).

Carlson et al. (2002) describe five mental actions that link covariation and rate: (1) coordination of variables; (2) direction of change; (3) coordination of the amount of change in the two variables; (4) average rate over uniform increments; and (5) instantaneous rate.

Gradient is a closely related idea that in school always relates to slopes of graphs at particular points. Connecting gradient to overall or instantaneous rate of change depends on having a dynamic view of the purpose of the graph (Gravemeijer & Doorman, 1999). However for linear functions instantaneous rate of change is indistinguishable from change on an interval and thus the gradient of the plotted line, whereas for non-linear polynomial and other school functions a formal understanding of gradient requires instantaneous rate of change. We reflected this in our survey by having a task that required formal knowledge of linear functions and asked about gradient, and a quadratic task in which varying rate of change is implicit.

Students' everyday knowledge of real-world events can serve as a basis for learning how to interpret a graph of a function (Goldenberg, 1987), but matching graphs to everyday life situations has been shown to raise particular difficulties. The most frequently cited difficulty is interpreting a graph as a literal picture of a situation (e.g., Clement, 1985; Janvier, 1981). Students deal more competently with graphs of functions when one of the variables is time or time-dependent (Janvier, 1981; Thompson, 1994a). The familiarity of time, plus its unidirectional nature (time only increases), seems to account for this. Students only need to apprehend how one variable (the non-time variable) varies; the time variable is taken into account implicitly. In our survey therefore we included interpretation tasks, not all with time on the horizontal axis.

The majority of studies focusing on covariation concern calculus students. Thompson (1994a), for example, investigated students’ reasoning on continuous covariation of quantities, one of which is time. Other studies focus on students’ early conceptions of covariation in the elementary grades, pointing at students’ capabilities and on possible pedagogical moves to support these capabilities. In a teaching experiment with 8 year old students, Warren and Cooper (2007) asked students to describe sequential growth patterns as functions. They found that tables of values appeared to draw students’ attention away from the role of the position number as the independent variable. Asking explicit questions that focus on the relation between position number and value, and avoiding sequential presentation of data, seemed to help students. We therefore included tasks that disrupted sequential representations. Confrey and Smith (1994) report that young children were able to develop the idea of 'unit per unit' comparison in situations involving realistic materials. Blanton and Kaput (2005) also suggest that it is possible for young students to express a naive form of functional thinking verbally and symbolically when they attend to the relevant connections. We therefore asked for verbal explanations of linear function situations. In similar studies using sequential data, students were found to tend to begin by examining differences in the dependent variable (e.g., Stacey, 1989) or improperly apply direct proportional reasoning (Stacey, 1989; Van Dooren et al., 2005). We provided tasks that would reveal these tendencies. Most relevant studies concern either undergraduates or primary school age children. Our research attempts to understand what happens between by concentrating on the secondary years (ages 11 to 18) that connect elementary knowledge to an eventual abstract understanding of functions.

**Adolescent learning of formal concepts**

Our empirical field (Dowling & Brown, 2010) is the secondary classroom and our focus is students’ activity in tasks related to functions. The starting point for our theoretical field is a sociocultural perspective. In studying cognition the majority of work in the field has taken perspective in which learning is seen the construction of knowledge by the individual mind on the basis of her/his actions. Berger (2005) explains that a drawback of this approach is that it is:

 ... rooted in a framework in which conceptual understanding is regarded as deriving largely from interiorised actions; the crucial role of language (or signs) and the role of social regulation and the social constitution of the body of mathematical knowledge is not integrated into the theoretical framework. (p. 153)

In the context of learning about functions, Dreyfus and Eisenberg (1983) point out the importance of relating students' everyday experience to the formal relations of an algebraic approach. We therefore take a Vygotskian perspective and look for the processes of mediation, complexes and pseudoconcepts, on the way to conceptual understanding (Vygotsky, 1978; Vygotsky, 1994), identified in large part as reaching conformity to the conventional body of knowledge, in this case of mathematics. Pseudoconcepts are particularly important in mathematics, since these are understandings that allow a student to align what they do and say with the activity of an expert, although they have not yet fully construed the concepts. We therefore focus on what students did and how they expressed it, comparing this to the curriculum and what teachers claimed they would be able to do, as these indicate how their classroom experiences were framed. Our study is therefore as much a study of the effects of teaching as it is of the progression of learning and knowing across the years of secondary schooling. In relation to the former, we are not able to look at the teaching directly, as it is beyond our resources to describe students' accumulated past experiences that have shaped current learning. Instead, assuming that the teaching is significant in the students’ learning we would need to maintain, as far as possible, similar teaching when examining what the students know as they progress through their schooling. The samples are therefore taken from two schools in which the same teachers, all of whom were intimately involved in the project throughout, teach throughout the range of classes as a proxy for controlling pedagogic form, which can never be achieved by researchers. We focus particularly on the tools for thinking that are introduced in the given curriculum (Newman & Holzman, 1993). By 'tools for thinking' we mean the iconic and symbolic representations, such as diagrams, symbols, layouts, language, that are used by teachers and students to express mathematical ideas, and the school conventions of their use. We therefore included tasks that used the conventions that students would have met in school.

In England generally, and in the schools we used, students generalize simple sequences, including linear, and use input-output flow diagrams by year 7. These flow diagrams are called 'function machines' and are used to solve simple equations or represent order of operations rather than to generate or represent functions, so the word 'function' is not introduced in its formal meaning. The English approach to functions has been widely influenced by an informal, qualitative approach developed in the 80s and 90s (Swan, 1980). Common experiences for students in England are 'pretend' contexts for functions, fitting lines to scattergraphs by sight, matching graphs to worded situations, and focusing on the qualities of graphs. These are alongside the more formal quantitative and algebraic approaches to graphing that progress via linear and quadratic examples (usually expressed as 'y = ') to a formal introduction in year 12, using definitions of functions and their properties and the f(x) notation, for those who choose advanced mathematical study. We would expect, therefore, that the effects of mediation through informal and formal language, i.e. mathematical conventions and symbols, on students' learning will be demonstrated in students' responses to tasks.

**Adapting constructs from Leinhardt et al. (1990)**

In their review of tasks associated with functions, Leinhardt et al. (1990) discussed the issue of task in functions, graphs, and graphing using four overlapping constructs: the action of the learner, the situation, the variables and their nature, and the focus. We used their framework to choose and characterize the tasks included in our study as well as to support, interpret and explain the findings. (Further elaboration of the way we used it can be found in Ayalon et al. IJSME)

*Action* refers to the kind of engagement expected.An action can be mainly interpretation, such as gaining meaning from a pattern, a table, or a graph, or construction, such as determining an algebraic expression from a pattern.

*Situation* refers to the setting and context of the task. In our case the setting is always the mathematics survey taking place in the usual classroom so we do not mention it further. The context for a task can be more or less contextualized or abstract thus influencing students’ approaches (e.g., Steele, 2008). We therefore broadened the meaning of 'situation' to include details about design and presentation that might influence approaches.

*Variables.* As discussed earlier, there are several meanings and aspects of a variable: contextual or abstract, continuous or discrete, compound or simple. These meanings are crucial to understanding functions and their representations.

*Focus* refers to the locus of attention. Depending on the situation and action, there may be several possible levels of focus in a task. A novice may focus on visual features of a graph, where a teacher may expect the focus to be on a particular analytic property of the function.

These four constructs informed us in devising the survey tasks, as elaborated below.

Informed by the ideas discussed above, this paper aims to present conjectures about:

* progression in understanding concepts towards functions, relative to curriculum and task features, across secondary years in one national context
* understanding variables: identification of and relations between variables
* relations between formal and informal understandings.

**Research Design and Methods**

**Sample**

Socio-economic backgrounds of the schools were similar and the four teachers with whom we worked, two from each school, all had mathematics degrees and mathematics teaching qualifications, and 4+ years' experience in the school. For this paper we focus on higher achieving students who are most likely to need to understand functions in later study or employment; these are classes who are generally taught, and expected to learn, more formal knowledge[[2]](#endnote-1). We are looking at the "best" possible range of responses that can be produced in those schools and therefore concerns which arise from them presumably do not have to do with a wide range of learning capabilities. This allows us to make conjectures about connections between concerns and task design, curriculum, teaching, and inherent difficulties of the concepts involved.

We needed a sample that was large enough to encompass a wide range of possible responses, while being small enough to analyse individual responses in detail and take the full range of responses into account. The survey was given to suitable classes from each of year 7 (age 11-12) to year 11 (age 15-16) and also to first and the second years of post-16 mathematical study. Each pair of teachers gave the survey to classes they taught from alternate years and then selected 10 from each class proportionally from alphabetical class lists. These were anonymised before handing them to us. Thus we received 10 survey responses (scripts) from each year 7 to 11 inclusive, and 10 scripts from the first and second years of post-16 mathematical study (total of 70 scripts).

**The Development of the Survey**

Tasks in the survey were devised over several design cycles in a process of conceptual investigation, starting with a hypothetical concept map of the development of the function concept in pure mathematics and in its uses for modeling phenomena. This map was further developed using the curricula, research literature as reported above, and teachers' expectations. Tasks were developed with the teachers to ensure they were compatible with language use and were appropriately accessible. We used tasks from Wilmot et al. (2011) and Swan (1980) as resources as these are based on the considerations above. The final design therefore drew on teachers’ knowledge and expectations of students, responses to trial versions, tasks familiar in textbooks and tests, teachers' reports of the order and content of their teaching, knowledge of curriculum, and from research. As well as working on the instrument, the same teachers also conducted the survey after they had all approved it, and provided insights about their pedagogy.

With reference to the teachers, the literature described above, and the framework from Leinhardt et al. (1990), this selection would capture a range of knowledge about progression towards functions without overburdening anyone. The five tasks we discuss in this paper could all be accessed via the curriculum in both countries. The sixth task depended on interpreting formal definitions of the word 'function', so was only accessible to older students in England and is therefore omitted from this paper. The tasks appear in the results section below so that students' responses can be directly compared to the task presentations.

**Data Analysis**

For each task we categorised all the different approaches students had taken according to the reasoning they described, whether successful or not. In cases where reasoning had to be inferred decisions were made collaboratively and consensually in the research team. Categorisations were constantly checked using the whole data set. Each response had to fit into one, and only one, category and this meant that categories sometimes had to be redescribed for the whole data set. This process required several passes through the whole data and an intimate knowledge of the data, made possible by the relatively small sample sizes. This process of creating categories from the data itself was necessary to generate as full a picture as possible of what students do in functions-related tasks.

**Results**

For each task we shall introduce the task and describe, using categories from Leinhardt et al. (1990), features of the task: our foci, the actions students would have to make, the situation, and the nature of variables. For this paper we constrain ourselves to foci relating to reasoning with variables, that is selection and recognition of variables of various kinds; expressing relations between variables, including correspondence, covariation, rates of change, gradient; and interpretation of relations presented as graphs, equations, and situations. Teachers' expectations follow each task description. We then present the categories generated in the analysis process, giving examples and quantitative results representing the distribution of the categories across years within the classes.

**Task 1**

 **Fig. 1. Task 1**

*Focus: Implicit and explicit rate of change, Relating dependent and independent variables.*

*Action:* Comparing both data columns to identify rate of change transforming it into 'floors per second'. Handling a non-sequential independent variable. Local action on particular values in 1.1; global action in 1.2.

*Situation:* Data presented in tabular form. The non-sequential data was designed to challenge a term-to-term tendency. 'Rate' has everyday meanings: "so much per so much". Context is a realistic position/time relation.

*Variables:* Contextualized variable, time as the independent variable, presented as if discrete; expression of constant rate of change expected.

*Teachers' anticipations:* Students of all ages would spot the correct missing value in 1.1, but as they did not teach rate they expected students to have some problems with 1.2.

Three categories emerged from students' expressions of rate of change, see Table 1.

Table 1

*Categories for Task 1*

|  |  |  |
| --- | --- | --- |
| **Category**  | **Description** | **Example** |
| **1** | Term-to-term (right hand column) | Q 1.1*[-2] because from 14 and downstairs the floors are going down in 4’s.*Q 1.2 (same student)[Four floors per second] *because the elevator graph shows that it skips 4 levels.* |
| **2** | Term-to-term, moving to covariation/rate | Q 1.1*It will be -2 because it is going down in fours so you just had to take away four from 2.*Q 1.2 (same student)*I think it is two floors per second because if you go to floor 12 it will be 1 second.* |
| **3** | Covariation/rate | Q 1.1*The lift will be at 0 after 7 seconds because its rate is 2 floors per second. It generally goes 2, 4, 6 but 7 is one second so zero is counted as a whole floor!*Q 1.2 (same student)*The rate is two floors per second because the floor number descends by 4: 2 seconds, so for every 2 floors: 1 second.* |

Table 2 presents the distribution of the categories of expressions of rate of change. As shown in Table 6, 64 out of 70 students were able to provide formal expression of rate of change as floors per seconds for question 1.2 (categories 2 and 3). A quarter of these responses were of category 2: In these cases, students chose the wrong answer of -2 when asked to find where the lift will be after seven seconds, explaining that the lift is going down 4 every second. However, when asked to find the rate at which the lift descends, they used both variables to get the correct answer. Despite the stimulus in question 1.3, there was no evidence that any rethought question 1.1. A much smaller number of responses showed a term-to-term approach throughout the whole task (code 1). Use of term-to-term declined with age.

Table 2

*Distribution of the Categories of Expressions of Rate of Change*

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Code**  | **7 year** | **8 year** | **9 year** | **10 year** | **11 year** | **12 year** | **13 year** | **Total** |
| **1** | 3 | 1 | 0 | 1 | 1 | 0 | 0 | **6** |
| **2** | 5 | 5 | 0 | 2 | 1 | 2 | 0 | **15** |
| **3** | 2 | 4 | 10 | 7 | 8 | 8 | 10 | **49** |
| **Total** | **10** | **10** | **10** | **10** | **10** | **10** | **10** | **70** |

**Task 2**

**Fig. 2. Task 2**

*Focus:* Generalise relations between discrete variables; relate data to spatial structure, avoiding assumptions about sequential patterns.

*Action:* Interpretation of a geometrical pattern and covariation or correspondence between number of hexagons and perimeter. Construction of a method of calculation and a corresponding algebraic expression. 2.1: Local and quantitative actions (calculating perimeter for given small number of hexagons). 2.2: Global and qualitative actions of structuring generalization, possibly finding rate of change. Translation between geometric, verbal, and algebraic expression, to function notation.

*Situation:* Data presented in geometric and verbal relational form. There is no table of values and data are not given in numerical order, to avoid term-to-term responses and encourage a focus on the correspondence relation. Questions 2.1 and 2.2 are intended to direct students towards understanding the relations and structure they need to express algebraically in 2.3. Context: artificial but 'realistic' discrete spatial growth situation.

*Variables:* Sequence number as independent variable. Discrete. Implicit use of constant rate of change expected.

*Teachers' expectations:* Students will be fairly competent and will use correspondence approaches since such tasks were familiar in school. They expected students to use input-output reasoning with the step size as the multiplier, although they might instead generalise directly from the geometric structure.

Two parallel analysis processes were implemented to describe approaches to functional reasoning and to classify the level of generalisation. Five categories of functional reasoning were found in students' attempts to find and express relations in the data, as presented in Table 3.

Table 3

*Categories of Functional Reasoning for Task 2*

|  |  |  |
| --- | --- | --- |
| **Category** | **Description of category** | **Example** |
| **1** | No answer, often “I don’t know”. |  |
| **2** | Empirical methods involving counting. | *You count how many edges in the chain* |
| **3** | Correspondence: Developing a general rule of the relation between the number of hexagons and the perimeter  | *2\*5, then how many hexagons in the middle of the end ones \* 4. Because if you had 3 hexagons together the end ones have five sides 5\*2 then the middle one has 4 so 1\*4 then add the answers together = 14.* |
| **4** | Covariation: Comparing or coordinating the two varying quantities | *4n+2. You can see the pattern of the perimeter value. They increase by 4 every time for every 1 hexagon! But start at 6 (for 1 hexagon)*  |
| **5** | Correspondence followed by covariation  | Q 2 *Get one hexagon and times it by a hundred*Q 3&4 (same student)*p(n) = n\*4. Because the perimeter increases by 4 for every 1 hexagon so it is number of hexagons multiplied by 4* |

The representation of generalisation from each student was also categorised to see which approaches were more successful: (1) no correct generalization of any kind; (2) generalization expressed correctly in verbal terms only, or (3) generalization expressed correctly algebraically as well as verbally.

Table 4 presents the distribution of the approaches to functional reasoning in students’ responses within the classes. As shown in Table 4, the most common approach (57%) to conceptualizing the functional relationships was the *correspondence* *approach* with students suggesting a general rule for the relation between the number of hexagons and the perimeter. The *covariation approach* of explicitly coordinating the two varying quantities while attending to the ways in which they change in relation to each other was less widespread (27%). Responses expressing a *correspondence approach* when addressing the question of finding the perimeter of 100 hexagons, and then *covariation approach* when generating a formula for any number of hexagons, constituted 14% of the responses.

The triples in the cells show the distribution of representation of correct generalisation: (none; verbal; verbal and symbolic). 35 students generated the correct generalization verbally and succeeded in forming it algebraically. Only 3 students verbalised a correct calculation without generating the corresponding algebraic expression. Results show progression towards full generalization across years.

The strongest connection between approach and successful generalization is with the *covariation approach*, which had a 79% success rate among those who used it (15 out of 19). The *correspondence approach* had a 51% success rate (20 out of 39)*.* Those who did not succeed with a *covariation approach* failed because they did not take starting values into account. Those who did not succeed with the *correspondence approach* mainly assumed proportionality. Note that the *correspondence followed by covariation approach* was always unsuccessful. Those who took this approach began with assuming a proportional relationship, and then moved to *covariation approach* based on step size without taking starting values into account.

Table 4

*Distribution of the Approaches to Functional Reasoning and* Representation of Generalisation

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Category** | **7 year** | **8 year** | **9 year** | **10 year** | **11 year** | **12 year** | **13 year** | **Total** |
| **1** | 0 | 0 | 1(1,0,0) | 0 | 0 | 0 | 0 | **1 (1%)****(1,0,0)** |
| **2** | 0 | 0 | 1(1,0,0) | 0 | 0 | 0 | 0 | **1(1%)****(1,0,0)** |
| **3** | 6(6,0,0) | 3(2,0,1) | 5(3,0,2) | 4(2,0,2) | 7(2,2,3) | 8(1,1,6) | 6(0,0,6) | **39 (57%)****(16,3,20)** |
| **4** | 1(1,0,0) | 5(3,0,2) | 1(0,0,1) | 4(0,0,4) | 2(0,0,2) | 2(0,0,2) | 4(0,0,4) | **19 (27%)****(4,0,15)** |
| **5** | 3(3,0,0) | 2(2,0,0) | 2(2,0,0) | 2(2,0,0) | 1(1,0,0) | 0 | 0 | **10 (14%)****(10,0,0)** |
| **Total** | **10****(10,0,0)** | **10****(7,0,3)** | **10****(7,0,3)** | **10****(4,0,6)** | **10****(3,2,5)** | **10****(1,1,8)** | **10****(0,0,10)** | **70 (100%)****(32,3,35)** |

**Task 3**

Fig 3. Task 3

*Focus:* Gradient of parallel straight lines; relating two representations of linear functions.

*Action:* Inferring from parallelism to using gradient. Reading the gradient in different representations, picking the gradient from the formal algebraic representation (the coefficient of the *x* variable) as well as from the graphical representation (using right angle triangles).

*Situation:* Formal presentation with no everyday options to use instead.

*Variables:* Abstract continuous variables. Gradient comparing changes in variables.

*Teachers' expectations:* Gradient and linear graphs are taught in year 9 so that students in and above that year would be successful; problems that would be introduced by having differently-scaled axes are avoided.

There were four categories of expression of gradient, as presented in Table 5.

Table 5

*Categories for Task 3*

|  |  |  |
| --- | --- | --- |
| **Category** | **Description** | **Examples** |
| **1** | No evidence of understanding gradient | [chose answer 4] *The line goes through 5 on the graph and the next point is 2 on the axis.* |
| **2** | Some procedural knowledge of checking whether lines are parallel but no use is made of gradient (instead – plotting and visual reasoning to show parallel lines). | [chose answers 1 and 3] *If y=2x+5 any equation that contains 2x will be parallel. And graph 3 because plotting the straight line y=2x+5 on the graph shows that it is parallel* [she plotted some points in the graph based on a table of values she built]  |
| **3** | Partial use of gradient in graphical terms as well as in algebraic terms. | *No. 1 is parallel because the amount of x’s determines the gradient. The +… = how high/low on the graph it is. No. 1 and y=2x+5 have the same amount of x’s. No. 3 is parallel because the formula of the graph is 2x so y=2x+5 is 5 higher but the same gradient.* |
| **4** | Relating gradient in graphical terms to algebraic terms. | *The gradient is the same it is just on a different y-intercept* [Used the right angle triangle in the graphs]. |

Table 6 presents the distribution of categories of expression of gradient. About half of the students from year 9 upwards expressed some use of gradient in graphical terms as well as in algebraic terms and this is progressive. About a half of the students overall (the proportion decreases with age) showed no evidence of using gradient. Partial use is shown mostly in the middle age bands (years 9-11).

Table 6

*Distribution of the Categories Related to Expression of Gradient*

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Code**  | **7 year** | **8 year** | **9 year** | **10 year** | **11 year** | **12 year** | **13 year** | **Total** |
| 1 | 10 | 8 | 4 | 4 | 2 | 5 | 1 | **34** |
| 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | **2** |
| 3 | 0 | 0 | 4 | 2 | 5 | 0 | 0 | **11** |
| 4 | 0 | 1 | 1 | 4 | 3 | 5 | 9 | **23** |
| **Total** | **10** | **10** | **10** | **10** | **10** | **10** | **10** | **70** |

**Task 4**

Fig 4. Task 4

*Focus:* Analytic features of graphs of quadratic functions.

*Action:* Noticing and communicating (formally or informally) features of quadratic graphs.

*Situation:* The first non-linear function students meet in formal representations is presented. Graphs can be treated as visual/pictorial/geometrical objects so responses could be about visual as well as analytical features.

*Variables:* Abstract variables; varying gradient; zeroes of dependent variable.

*Teachers' expectations:* Year 9 and onwards plot quadratics and could highlight analytical features formally.

The task as a whole was coded using four categories:

1. No reference to features at all.
2. Reference only to visual features (e.g., thin, high, low) that do not express reasoning about variables or functions.
3. Reference also to (at least one) analytical features (e.g., orientation, transformation, turning point, zeroes) but with no formal language.
4. Reference to (at least one) analytical features using formal language of functions.

Table 7 shows examples of some of the analytical features and how they were indicated, informally or formally.

Table 7

*Examples of Features in Task 4*

|  |  |  |
| --- | --- | --- |
| **Analytical feature** | **Formal / Informal** | **Example**  |
| Orientation | Informal | *B and C look like mountain way up but A is not*  |
| Orientation | Formal | *Both B and C have maxima* |
| Gradient | Informal | *A and B have the same gradient as each other except B is negative* |
| Gradient | Formal | *The magnitude of the gradient functions is the same* |
| Turning point | Informal | *The top of the arch both contain 2 on the x axis* |
| Turning point | Formal | *Both B and C's optimum point is at 2 on the x axis* |
| Zeros | Informal | *They both touch the point (2,0)* |
| Zeros | Formal | *A and B intercept the x-axis at two points so have two real solutions, whereas graph C only has 1* |

Table 8 presents the distribution of categories related to levels of interpreting graphs of quadratic functions across years. As can be seen, there is a “jump” in looking at the graphs of the functions analytically from year 8 to year 9. In years 7 and 8 students mostly attended to visual features of the graph. In contrast, from year 9 and onward students exhibited an analytical approach to the task (categories 3 and 4). Features of orientation and zeroes received most attention throughout all years but for reasons of space we do not show this data. Using formal language begins very slightly in class 10 and increases (category 4). It is moderately used by students in years 11 and 12 and highly in year 13.

Table 8

*Distribution of Levels of Interpreting Graphs of Quadratic Functions*

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Code**  | **7 year** | **8 year** | **9 year** | **10 year** | **11 year** | **12 year** | **13 year** | **Total** |
| 1 | 2 | 4 | 0 | 0 | 0 | 0 | 0 | **6** |
| 2 | 4 | 2 | 0 | 1 | 0 | 0 | 0 | **7** |
| 3 | 4 | 4 | 10 | 8 | 6 | 6 | 2 | **40** |
| 4 | 0 | 0 | 0 | 1 | 4 | 4 | 8 | **17** |
| **Total** | **10** | **10** | **10** | **10** | **10** | **10** | **10** | **70** |

**Task 5**

Fig 5. Task 5

*Focus:* Identifying variables and relations between variables; covariation. Qualitative understanding of covariation and how it is represented by graphs. Relating two representations: verbal situations and graphs.

*Action:* Interpretation task: matching graphs and verbal situations. Global qualitative: relate entire graph (or part of it) to the situation; identify variables and relations between them; patterns of covariation; contextual features and zeroes. Translation between representations.

*Situation:* Realistic contexts. Axes are included for students to label variables to help them be explicit about variables and relations.

*Variables:* Continuous, in four situations: i. time, volume of noise; ii. price, profit; iii. time, price or price rise; iv. speed, time (as dependent variable). Compound variables in situations iii: price rise, & iv: speed. Time is not always the independent variable. There is a choice of variables in situation ii. Covariation qualities: increase/decrease; variable rate of change; leveling off.

*Teachers' expectations:* Students from year 7 onwards meet real-life graphs of different descriptions (e.g., conversion graphs, distance-time graphs, volume-time graphs); there would not be much variation in response throughout years 7 to 13 or between situations.

Analysis of the 70 x 4 responses resulted in three codes:

1. No choice, often accompanied by “I don’t know”.
2. Lack of full analysis, further categorised below.
3. Full analysis.

Further analysis of code 2 showed four sources of difficulty, which we shall exemplify in Table 9.

Table 9

*Difficulties in Task 5*

|  |  |  |
| --- | --- | --- |
| **Code**  | **Sub-category** | **Examples of student response** |
| **2a** | Focus on one variable with picture/graph confusion  | Chose *j* for situation ii: *The stages represent the prices of the movie tickets* [marked three places on the graph: when the tickets are low, high, staying in the middle]. |
| **2b** | Choosing one relevant and one irrelevant variable forming an irrelevant relation | Chose *g* for situation ii: *Because people will still pay so it is slowly getting better*. [wrote time as (*x*) and profit as (*y*)].  |
| **2b** |  | Chose *f* for situation iv: *It represents a person running very steadily and slow* [wrote “start” at the high point and “finish” at the low point]. |
| **2c** | Choosing relevant variables but forming an inadequate relation between them  | Chose *e* for situation iii: *Price is rising but started slowly* [wrote time as (*x*) and price as (*y*)].  |
| **2c** |  | Chose *c* for situation iv: *The higher the speed the lower the time and this graph shows that* [wrote time as (*x*) and speed as (*y*)]. |
| **2d** | Failing to notice contextual features  | Chose *d* for situation i: *As more people joined in the people around them did as well so exponentially increasing the cheering* [wrote time as (*x*) and cheering as (*y*)]. |

Table 10 presents the distribution of categories related to level of analysistaking all four situations into account. There is identifiable age-related progress towards full analytical interpretation. In the early years most responses reflected an inability to approach the task (code 1) and as years progressed responses of full analysis became more apparent (code 3) but the occurrence of code 2 was similar throughout all years. The quartets in the cells show the distribution between the situations 1–4, respectively.

Table 10

*Distribution of Level of Analysis (All Four Situations Taken Together).*

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|   |  | **7 year** | **8 year** | **9 year** | **10 year** | **11 year** | **12 year** | **13 year** | **Total** |
| **1.**  |  | 26(6,5,7,8) | 33(5,9,9,10) | 11(5,2,1,3) | 7(0,4,0,3) | 8(1,3,1,3) | 3(0,1,0,2) | 3(0,0,1,2) | **91****(17,24,19,31)** |
| **2.**  |  | 14(4,5,3,2) | 6(5,1,0,0) | 19(5,4,5,5) | 22(8,4,5,5) | 16(5,3,4,4) | 17(7,3,2,5) | 12(6,2,0,4) | **106****(40,22,19,25)** |
| **3.**  |  | 0(0,0,0,0) | 1(0,0,1,0) | 10(0,4,4,2) | 11(2,2,5,2) | 16(4,4,5,3) | 20(3,6,8,3) | 25(4,8,9,4) | **83****(13,24,32,14)** |
| **Total** |  | **40** | **40** | **40** | **40** | **40** | **40** | **40** | **280****(100%)** |

Subcategories 2a, 2b, 2c, and 2d constituted 5%, 22%, 33%, and 40% of all the responses, respectively. These findings show the criticality of identifying variables and how they covary (2b, 2c, 2d), rather than the picture/graph confusion (2a) prominent in the literature. There was considerable variation in the frequency of these difficulties between situations: 100% of responses associated with category 2a appeared only in situation ii, mainly picking price as being ‘high’ or ‘low’ (see example in Table 5, first row). 84% of responses associated with category 2d appeared in situation i, not noticing saturation (see example in Table 5, sixth row). Responses associated with category 2b appeared in two situations: 65% in situation ii, plotting *profit* against *time* instead of *ticket price* (see example in Table 5, second row), and 35% in situation iv, plotting *distance* or the *distance left* as related to *time* (see example in Table 5, third row). Most responses associated with category 2c appeared in two situations: 49% in situation iv, making a linear assumption (see example in Table 5, fifth row), and 34% in situation iii, making a linear assumption or choosing the wrong way up (see example in Table 5, fourth row). Sources of difficulties appear to be associated with some situations and not with others. They also persist across years.

**Synthesis of Progressive Changes**

Analysis of students’ responses revealed progression towards correct answers as we would expect, but not always in line with teachers' expectations and the curriculum. However, our analytical approach was to identify how students appeared to reason and not only how correct they were. This approach enabled us to observe several key understandings that appeared to be progressive as well as others that appeared not to be.

In years 7–8 there were tendencies to use term-to-term and proportional assumptions when working on the sequential tasks 1 and 2, respectively. These behaviors are well known in the literature (e.g., Stacey, 1989; Van Dooren et al., 2005), and both tasks were constructed with features intended to disrupt these tendencies. In years 9–11 there was progression in overcoming these tendencies, which totally disappeared in years 12–13.

Evidence of reading the gradient in symbolic algebraic representation and graphical representation (Task 3) appeared slightly in years 9­–11 with most students treating gradient informally as a visual slope or as the coefficient of x. Formal understanding becomes apparent in years 12–13, long after it was first taught in year 9. Students in years 7–8 showed no evidence of understanding a relationship between gradient and parallelism, as expected. Attention to analytical features of quadratic functions represented in graphs (Task 4) improved across years. Students in years 7–8 attended mainly to visual features, sometimes picking out features that are analytically important, students from years 9–11 and even more so in years 12–13 noticed analytical aspects, with variety increasing with age. The use of formal language appeared slightly in years 9–11 and increased in years 12-13 but not to a large extent. In Task 5 identifiable age-related progress was found towards full analytical interpretation, including identification of relevant variables and forming the relation between them that reflected the verbal description, although teachers had assumed there would be no age differences.

The above two paragraphs highlight key aspects that appeared to progress. Now we turn to emphasize several aspects that did not vary across years. One aspect is related to students’ capability in finding the formal rate of change in Task 1. Almost all students, from all years, successfully identified the rate of change, including those who at the beginning used the term-to-term approach. This is not trivial as teachers informed us that the concept of rate is not introduced in school until the 12th year, and they did not expect such success in years 7 and 8, and the use of the word 'rate' is not necessarily enough to guarantee correct reasoning (Herbert & Pierce, 2012).

A second aspect concerns approaches used by students in Task 2 (e.g., correspondence, covariation, correspondence followed by covariation). The numbers of students who used each of these approaches was similar across ages. Although results had indications of progression towards successful algebraic generalization, a specific age did not seem to be linked to a specific approach. The use of the correspondence approach across years might not be a surprise, as according to the teachers this is the approach generally presented in classes, but many of the correspondence approaches involved a proportional assumption, so could be more aligned to a 'missing value' approach than recognising a linear function (De Bock, Van Dooren, & Verschaffel, 2005).

The third aspect appearing not to change across years relates to the types of difficulties encountered by students when prompted to match realistic situations to graphs (Task 5). These difficulties were found to be mainly related to identification of relevant variables and their relations, but do not follow the hierarchy that we expected. These difficulties did not vary between ages, but did vary between specific situations. Thus, similarly to Task 2, while students became better at overcoming these difficulties, there was no fading of the difficulties themselves.

**Discussion**

This paper set out to identify progression in expressions of reasoning about variables across five tasks in two English schools. Key ideas were identifying variables, forming relations between variables - correspondence and covariation - and related aspects of rate of change and gradient, students’ capabilities, and difficulties. This section begins with highlighting some important findings related to reasoning about variables. We then offer conjectures concerning links between the affordances of the tasks, the curriculum, and students' responses, and about the formality of understanding and expression. Finally further research to extend and strengthen our findings is proposed.

**Reasoning about variables**

Throughout tasks 1, 2, and 5, we hoped that students would be attentive to the ways in which two quantities change in relation to each other in the various contexts and representations given, and would be able to identify and construct the variables and their covariational relation within a representation and between different representations (Carlson et al., 2002).

Students from all ages succeeded in finding the rate of change from a table of values (task 1) although the teachers reported that they have not met rate in class. Even those who began with term-to-term reasoning moved to a covariational approach when asked to give rate of change and therefore had to relate the 'gap' to changes in the independent variable. In task 2, almost a third of the students (from all ages) attended to data given in a geometric pattern using covariational reasoning, that is they were explicit about comparing changes in numbers of hexagons and perimeter, and most of those then succeeded in building the algebraic generalization. This is rather surprising as the teachers reported that they emphasize the correspondence approach in class and also because the way the task was set – not presenting data in numerical order – does not encourage a covariational approach.

These findings indicate that in linear situations even the younger students are capable of expressing covariational reasoning in terms of coordination of variables, expressing the direction of change, and coordinating the amount of change in the two variables, which are the first three levels in Carlson et al.’s (2002) framework. Similar capabilities were reported in other studies as well (Blanton & Kaput, 2005; Warren & Cooper, 2007). However, these studies were intervention programs focusing on covariational reasoning whereas students in our study did not have any special teaching and indeed their responses confounded teachers' expectations. Our findings suggest student capability with linear covariation even when such an approach is not explicitly required.

Reasoning about variables is fundamental in task 5. Interpretation of non-linear situations and graphs requires identifying the variables and thinking about how they inter-related, including coordinating variables, imagining the direction of change and coordinating the amount of change. Attention should be paid also to change over short intervals and in some situations also to instantaneous change – higher levels of reasoning (Carlson et al., 2002). Students have to override immediate reactions such as preferences for time or linearity (Janvier, 1981; Leinhardt et al., 1990; Thompson, 1994a). Instead, they have to make an effort of imagination to identify and coordinate variables in imagined situations. In contrast to Task 1, in which the realistic context presented a linear relation, task 5 gave non-linear relations requiring students to handle varying rates of change.

Frequent errors were choosing a non-relevant variable (usually time) and forming a non-relevant relation (usually linear), with types of difficulties varying according to situations. In situation i the only difficulty students encountered was failing in noticing contextual features of the situation, i.e., not noticing that there was a period when all were applauding. Students generally chose the relevant variables and formed their relation. This can be explained by the fact that the independent variable is time, so a movement from left to right across the axes can represent rate as change in quantity; without a need to understand instantaneous rate; the language did not express a need for rate of change. In situation ii the main difficulty students encountered was in choosing one relevant and one irrelevant variable (time) and then forming an irrelevant relation. In this situation the relevant independent variable is not one normally used in school, unlike time. Profit being higher or lower matters, but there is no reason why school students should have everyday knowledge to think about the marginal effects of changing ticket prices, i.e. the instantaneous rate of change. In situation iii students succeeded in identifying the relevant variables but encountered difficulty in forming the relation between the variables. In most cases it was about building the “wrong way up” relation, starting with slow rising of prices and continuing with faster rising. Some others chose a linear graph. This may be related to the fact that in this situation time is an issue, but the option is to plot prices so that rate of change is the gradient of the graph, and attention to instantaneous change, or change over short intervals, is a key feature of the situation. None of the students chose rate of change itself as the dependent variable, possibly because the covariation would then be the *second* derivative and hard to imagine.

In situation iv students’ main difficulty was in forming the relation between variables: choosing useful variables but forming a linear graph with negative gradient. The second main difficulty was in focusing not on speed-time, but rather on the travel of an individual, with the *distance* or the *distance remaining* as related to *time*. The words of this situation present an everyday idea of rate as 'distance over time' (slowest, longest) and students have to transform this into a scientific concept. The covariation required to be successful is not between distance (which is constant) and time but between rate (speed) and time in an unusual way. Students may be used to graphs which show variation in speed over time, but this situation requires a speed-time graph whose instantaneous rate of change is the marginal time benefit of increasing speed. In this situation it might be unhelpful to use formal knowledge about speed-time graphs and instead focus on global change in terms of 'where is it higher? Where is it lower?' The understanding required for such graph-matching tasks is therefore a qualitative understanding of covariation and how it is represented by graphs.

Overall, older students were showing the same range of errors and difficulties as younger students. We conjecture from this that students’ capabilities with the relationship between verbal and graphical descriptions of phenomena are influenced more by features of the phenomena than by formal teaching or maturity. However, the English curriculum which includes specific attention to applications appears to have a positive effect on students’ progression, i.e. students can be schooled towards being more capable of making the connections. Students’ experience seems to be more relevant than maturation: if it was maturation students would not find these difficulties by the time they get to calculus courses, as is found in research (Carlson & Oehrtman, 2005; Monk, 1992; Monk & Nemirovsky, 1994; Nemirovsky, 1996).

**Relating students' treatment of variables with aspects of design**

As declared in the beginning of the paper, our use of the word 'covariation' includes informal, everyday, qualitative senses of covariation, and also anything that indicates a comparison of changes in variables, whether this is over time or over some other interval, or instantaneous. In this discussion so far we have used the terms 'informal' 'formal' and 'everyday' in this and in other contexts. However, these terms do not fully capture the range of responses in our survey, nor our design intentions. We now reflect on the affordances of the tasks and the responses in terms of progression towards functions.

In task 1 we deliberately set out to use students' everyday understanding of elevators and their school-taught use of data tables. Since data tables provide a schooled format for organising quantitative information they could be described as 'formal'. However, they are not concepts in themselves in the Vygotskian sense; rather they scaffold attention towards plausible patterns in the data. Nor are they formal from the perspective of higher mathematics, in which the concept of function is well defined and associated with specific symbolisation. On the other hand, they are not everyday tools, because when we are using an elevator we do not use data tables to conceptualise our journey. Data tables are one of the many tools we have in mathematics, and in mathematics teaching, to structure information in ways that make it possible to think in terms of relationships and properties that were expressed more formally when we asked for 'rate' in 'floors per second'. Because 'rate' has everyday meanings (Herbert & Pierce 2012), we assume that the language form 'floors per second' is what triggers the correct answers, and this would reflect research results from Confrey and Smith (1995).

In task 2 we offer diagrams, and then ask for words that appear to shift students towards a covariation approach. The hexagon task is not an everyday experience, but we expected students to use school experience, which is of correspondence approaches in spatial sequence tasks, and also many brought inappropriate school experience of proportionality. From information we have about the teaching they have had, and from research about learning algebra (e.g. Radford, 2000), we expected that verbalisation would structure information about a construction process (i.e. an imagined action), and this connection was made by half the students.

In Task 2, we wondered why so many students took a covariational approach when the data was presented in a correspondence format, connecting input to output and not in sequence. Either students needed to transform the data so they could treat it this way, as they were used to sequential data, or they had internalised covariational reasoning as a powerful way to approach such tasks in general, even when data was not presented in an encouraging form in Task 2. In Task 1 we chose to give a familiar representation of data in tabular form to support students' identification of term to term changes, but this format also led many to make the expected error of assuming a vertical linear sequence. The format here enabled access, structured their thinking about what 'rate' means, and eventually overrode earlier incorrect assumptions. In both of these tasks, we had taken formats students would find familiar, but included unfamiliar features to see if they could attend to underlying meaning rather than habitual responses.

In task 3 we offered no similar possibilities; students who could not use the formal presentation had no everyday options to use instead; they cannot even use their spatial understanding of 'parallel' unless they understood the given equation. We could not design a way to scaffold students' performance in a task that depended on understanding the symbolic systems being presented. In task 4, we made sure that there were several visual features that could be used, so that students could treat the graphs as visual objects (as younger students tended to do) with no underlying analytical meaning, or could use the task to demonstrate analytical knowledge by choosing features which are particularly important with functions. A few older students used the word 'gradient' spontaneously. We offered formal versions at the end of the task, but no one used these.

In task 5 the use of everyday experience is an overt intention, and there are no formal options. To identify variables requires students to imagine the situations – which were easier in time-dependent cases – and think in terms of covariation. We had provided empty axes as a structuring device, as well as to show us the chosen variables. There is no formal understanding of functions required, but there is a need to understand the formatting power of graphs. Graphs not only format data and experiences that we already have, but also suggest structures from which we can interpolate data that we do not have. We offered scaffolding in the form of axes to support identifying variables and thinking about how they inter-related, we did not provide formatting that would support identification of rates of change.

Tasks 4 and 5 both offer progress towards an understanding of functions integrating formal and modeling purposes. Task 4 structures a connection between visual representation and the analytical properties of functions – a connection that strengthens as students learn more. As it stands the task does not make a strong connection between representation and the formalisation of such properties. Task 5 offers a strong connection between everyday experience, the analysis of variables and covariation within experience, and its graphical representation. Like task 4, but for different reasons, it does not offer a strong connection between representation and any formal knowledge of functions.

**Formality**

This reflection on the tasks suggests three interconnected layers of understanding: informal, schooled (including diagrams, formats, layouts and speech), and formal. In this way we can distinguish between the formal understanding of functions which depends on definitions and conventional symbolism, and understandings that have to be taught and learnt but are not, in the Vygotskian sense, academic or scientific concepts. That is, they depend on the symbolic tools that are used in school to support concept formation, and the familiarity of forms of speech and representation.

It does not follow that we can say that any individual student is or is not displaying informal or formal understandings, nor whether they are displaying concepts or pseudoconcepts. When they are using the required format we cannot say whether they have an underlying sense of structure or not. In task 3 we attempt to get at a specific manifestation, namely the connection between the way symbol systems present gradient and the concept of parallelism. However, we do not know from this task whether students understand gradient as a local rate of change, or a 'number in front of *x*' or as a description of global slope. For this reason we emphasise the need for special pedagogic attention to use of formatting tools that 'school' a structured understanding of function-related concepts and pseudoconcepts, both by teachers and students. We also propose avoiding dichotomies such as formal/informal that imply a straightforward difference. Some of this work is already being done in our field in relation to visualisation, diagram, talk and dynamic digital tools but it needs to be extended to other conventional formats such as tables, grids and coordinate axes as shown here.

Initial analysis of data from the other country, Israel, has influenced some the above discussion by pointing us towards phenomena that varied or were similar between countries. The value of comparisons with different curricula backgrounds is to suggest distinctions that might be due to curriculum rather than maturation, teacher knowledge, and so on. This work has and will be reported in more detail elsewhere (refs so far) but for this paper it may be helpful to know that the Israeli students, who have a more formal introduction to functions from a younger age, were much more successful in tasks 1 and 2, even given the heavy focus on mathematically equivalent tasks in the English curriculum. This led us to focus in detail on how English students had responded to the format and layout of the tasks, as evidenced in their responses, and the fact that they had to adapt their reasoning to fit the disruptions offered in the task, where the Israeli students already had some formal teaching about rates to draw on. Another major difference between the countries was in Task 5, where English students showed progression through school years, but Israeli student findings were more or less similar across ages. Within these different patterns, both sets of students showed the same kinds of item-related differences, suggesting that the difficulty was in the items and not the students' maturation or knowledge. This led us to focus on the variables, the relations, and how they were presented.

**Conclusion and Next Steps**

In tasks about typical school curriculum content that moves towards functions understanding, mathematical foci emerged as areas of difficulty, success, and progression beyond what is structured by the curriculum. These were in line with our initial premise: that identification of and reasoning about variables and the related concepts of rate of change and gradient are key contributing concepts in adolescent understanding of functions.

As well as the task-specific findings described above, we presented three insights that appear to apply to adolescent learning about functions:

* students were capable of coordinating variable change using schooled formats in linear situations, and this capability sometimes surpassed teachers' expectations based on the curriculum;
* qualitative understanding of variables, covariation and their representations is situation specific;
* coordination of informal and formal understanding, and use of informal and formal expressions related to covariation depends on experience over time and schooled formats as well as on curriculum structure.

Importantly, we have identified connections between informal, schooled and formal aspects of reasoning about variables in adolescent learning. Fine-grained focus on the effects of tools for thinking, i.e. the formats provided by curriculum materials and digital tools, is of practical importance and also pertinent when thinking about adolescent learning about functions in school.

The role of functions in the school curriculum is twofold: supporting learning in pure mathematics, and also in applying mathematics through modeling in a range of disciplines. Our study has provided a window through which to imagine students' capabilities within a teaching approach that provides some everyday graphing experience, and some formal algebraic experience, but does not formalise functions, and only links algebraic to everyday experience in simple situations.

Clearly our study has limitations, largely a result of the resources available to us. However, the small sample has allowed us to probe individual responses deeply and creatively. We repeat the work reported in this paper in the context of Israel, where the curriculum context is very different, with formal teaching of functions taking place from age 12 onwards, and expect to publish results of this comparison (ref to RME paper forthcoming). This will add depth to our understanding of the role of formal teaching about functions which takes place earlier in Israel than in England. The internalisation of concepts is carried out in classroom interaction, with classroom formatting tools, and we would expect that classroom studies could usefully extend our elaboration of informal, schooled and formal functions thinking.

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1. We are aware that the terms 'intensive' and 'extensive' are often used for this distinction, but find that these do not make always make intuitive sense for teachers. Compound variables, according to our use, are usually rates but we want to reserve the word 'rate' for our discussions about understanding. [↑](#footnote-ref-1)
2. We also surveyed the same number of lower attaining students but their responses were all informal so do not contribute to this paper. [↑](#endnote-ref-1)