UNIVERSITY OF BIRMINGHAM

Little Books of Adapting and Extending

Book 1: Isosceles Triangles

Pat Perks and Stephanie Prestage

UNIVERSITY OF
BIRMINGHAM

School of Education
University of Birmingham
Edgbaston
Birmingham
B15 2TT

Pat Perks p.a.perks@bham.ac.uk
Stephanie Prestage s.a.prestage@bham.ac.uk

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Welcome to this little book of ideas for use in classrooms from Key Stage 2 through to Key Stage 4

The purpose of the book is that you dip in as you wish, there is no order. Each task is defined as we might use it in the classroom. There are adaptations which allow the task and the mathematics to develop over the time of the task and/or the topic. And then there are some extensions depending on how deep you want the mathematics to progress and on the challenges your class can handle. You make choices - start anywhere in the book and at any point on the double page spread

The activities in this book are related to aspects of geometry connected to the isosceles triangle. It has been amazing for us to discover how much geometry can be developed from the isosceles triangle and can be deduced from knowing about the isosceles triangle.

Pat Perks: Pat has worked as a teacher of mathematics in secondary schools in Birmingham, as a head of department and as an advisory teacher. She currently works as a lecturer in mathematics education at the University of Birmingham.

Stephanie Prestage: Stephanie has worked as a teacher of mathematics in secondary schools in London and also as a head of department. She currently works as a lecturer in mathematics education at the University of Birmingham.

Contents

Introduction

Welcome to this draft version of little book of ideas for use in classrooms from Key Stage 2 through to Key Stage 4. We hope you will find something to use in your classroom.
The purpose of the book is that you dip in as you wish, there is no order. Each task is defined as we might use it in the classroom – by the way, the title of the task is for you and not the students. Then there are adaptations which allow the task and the mathematics to develop over the time of the task and/or the topic. And then there are some extensions depending on how deep you want the mathematics to progress and on the challenges your class can handle. You make choices - start anywhere in the book and at any point on the double page spread (we have left you space to add in your own changes to the tasks and any notes as you use them with classes).
We also list some of the mathematics which might happen through engaging with the task, you might want to look at this list to help you match up with your own departmental curriculum demands.
Below we give some of our philosophy about teaching and learning that lie behind this book. It comes from many years in the classroom (both of us were head of department) and many years working with student-teachers observing and talking about literally hundreds of mathematics lessons. Our thanks to all these teachers.

Adapting and extending

Several years ago we published a book entitled *Adapting and Extending* (Prestage and Perks, 2001) in which we set out ways of creating tasks for the mathematics classroom. The rhythm of the task-changing goes something like this:

find a task
↓
highlight the givens
↓
change the givens
↓
consider the new tasks

You can read the book to find out more about this task-changing routine. You might pick up the rhythm as you read the ideas in this book. Ultimately we hope that you can begin to invent your own tasks by making adaptations. As well as new tasks for another day you can also be ready with an adaptation in the moment when you need extra work in the classroom.
In the book we offer tasks as we might use them with some adaptations and extensions. Timing and choices will be dependent on the class of pupils you are working with.

Deep progress

A second philosophical and pedagogical standpoint is that learning is best if it is deep learning (Watson et al 2003), learning when I know something from having played with it from different perspectives and had lots of practice. I know the mathematics because of these experiences and not because I remember an algorithm. Teaching stays with a topic over time (four to six weeks) so that the mathematics and hence learning can progress making connections to earlier learning and preparing new learning for another day. It is possible. If you take the curriculum to have it main end point as GCSE - that leaves 5 years for learning and progressing in secondary school, and 9 years if you include junior school students. Go and look at a GCSE paper and then know this is the end point for 16 years olds. You have lots of time to map out progression for deep progress prior to this moment. It is also nice to save some 'new' maths for each year group or to teach beyond the topic so that questions for assessment seem relatively straightforward because you know the maths so well.

The isosceles triangle

The activities in this book are related to aspects of geometry connected to the isosceles triangle. It has been amazing for us to discover how much geometry can be developed from the isosceles triangle and can be deduced from knowing about the isosceles triangle.
There are two jobs for the learner. One is to learn about the relatively straightforward definition of the isosceles triangle with its properties of equal sides and equal angles and its most powerful line of symmetry, both angle bisector and perpendicular bisector. Lots of tasks in this book offer a context for playing

with these ideas. Then, knowing about the isosceles triangle, lots more geometry can be deduced.
Why is this shape so useful? What are its connections? How does it provide useful tasks for the classroom when what follows does not look like questions on examination papers?
Take the following example:

Draw a circle and a chord in the circle. Join a diameter to the midpoint of the chord. Join the end of the diameter at the circumference to the ends of the chord.

If you are in Key Stage 2, or the beginning of Key Stage 3, I might ask you to measure all the angles and sides and to repeat the task several times ready for a discussion about isosceles triangles.

- The task provides lots of practice of measuring angles.
- As the teacher, I know which angles should be the same I and that they sum to 180º so marking is straightforward.
- Because each pupil will get something different, there is a purpose for them checking each others work.
- They also get practice drawing circles.
- There will also be an opportunity to discuss accuracy and error.
- Pupils get the chance to set their own work.

In Key Stage 3 I might ask you to explain why the two right angled triangles are congruent.

- Pupils will be working from different diagrams so that checking congruence moves from individuals being able to measure angles and lengths in each pupil's special case to generalisation for all cases.

In Key Stage 4 I might ask you to explain why the diameter meets the midpoint of the chord at right angles

- Again there is the opportunity for many special cases with the move to generalisation.

The power of the isosceles triangle is that its geometry works in much of the more complex geometry met in key stage 4. The triangle is essential to cyclic polygons. It is the building block for proving the angle in a semi-circle is a right angle. It also provides special cases for other circle theorems. For us, the work with the triangle has changed our view of how to do the constructions. It is no longer a matter of remembering a list of rules, but thinking of the isosceles triangle. It also offers a real mathematical context for using symmetry.

The structure of the book

Each of the double page spreads has a task defined with some possible adaptations and extensions.

Tasks
Some of these may look very different from traditional text book questions, which tend to offer questions similar to those on examination papers, but they will provide plenty of practice of the content in the school syllabus and an opportunity for discussion. They also provide opportunities for pupils to use geometrical reasoning with justification, explanation and proof.

Adapting
The adaptations offer opportunity for lots of practice whilst working to different goals. Sometimes small adaptations are intrinsic to the original task. In the example just discussed, each pupil will have a different adaptation of the given diagram, lots of special cases. In other tasks there will be prompts to create more special cases as in C7:

> *Draw axes and draw some isosceles triangles so that $x = 6$ is the line of symmetry.*

Pupils have the opportunity to practice plotting and demonstrating their understanding of the line of symmetry and its relation to the isosceles triangle.
Different adaptations happen when the givens are altered – such as the equation of the line of symmetry in the above. You may want to do more of these than we suggest in the text with some classes.
As we have been writing so we have been adapting, as a result you will find some tasks have a lot in common with others. This is deliberate as we feel the adaptations change the mathematical opportunities.

Extending

With some classes, you may not use these. With others, this is where you may want to start. For yet others, these ideas may be used with only some of the pupils.

Notes:

1. Throughout the book we use the language equal sides and equal angles and the odd side and the odd angle for the different sides and angles of the isosceles triangle.
2. We think there are two types of triangle, isosceles and non-isosceles. The equilateral triangle is a special form of the isosceles triangle.
3. Much of the language in geometry curriculum is derived from Greek words. Isosceles is made up of two Greek words *ίσος/isos* meaning ‘equal’ and *σκέλος/skelos* meaning ‘leg’. Cyclic comes from the Greek word for a circle, *κύκλος*/*kiklos*.

References

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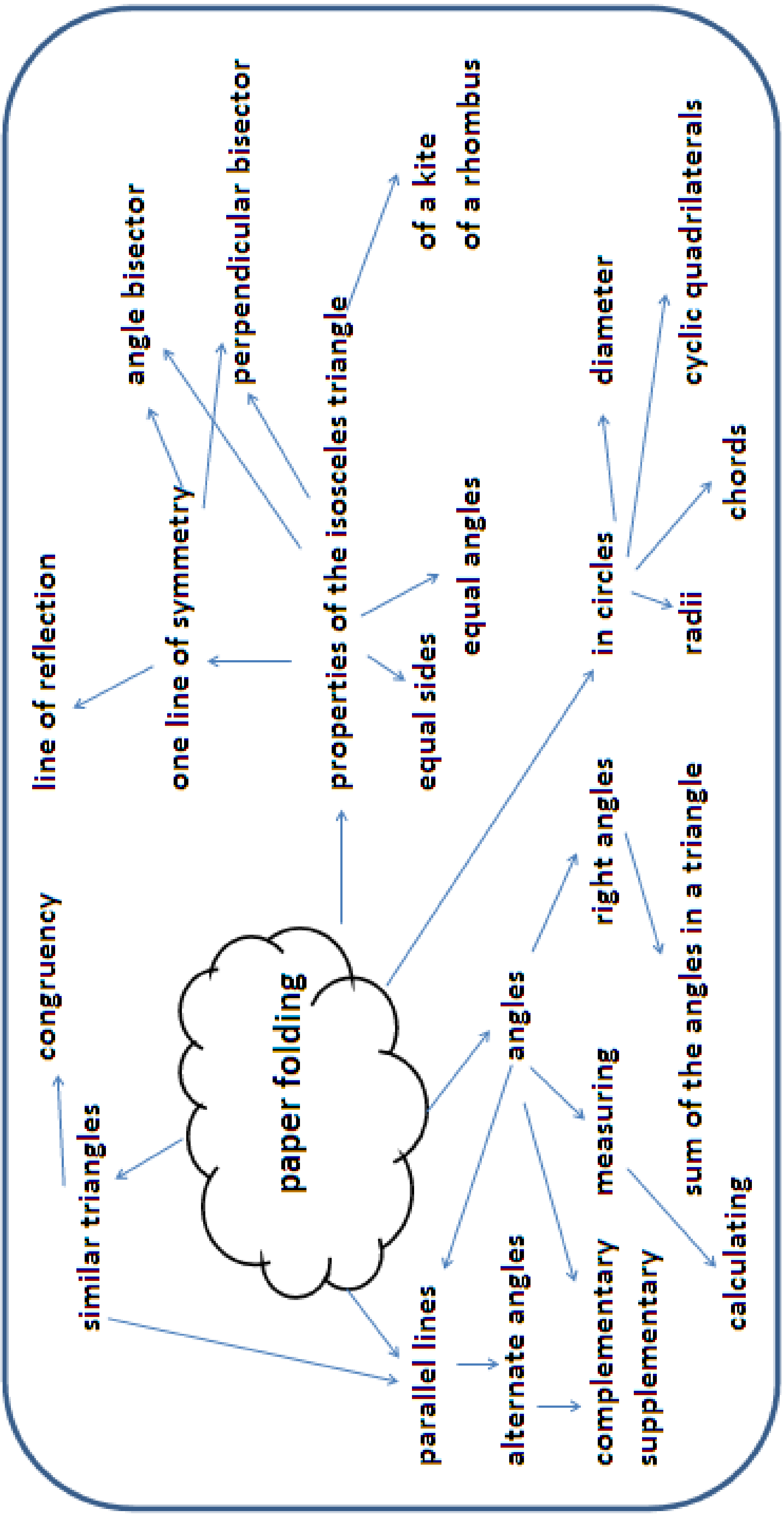
Section A: Paper Folding

Section Notes

The ideas in this section can be used with students throughout Key Stages 2, 3 and 4. When you are ready to play with some geometry these, folding paper tasks will hook the students into the mathematics and give each student their own physical copy of the work from which they can measure lines and angles and compare with others

We offer ideas to work on the isosceles triangle using scraps of paper, A4 paper and filter paper (this you can find in educational catalogues, in the science section) or you can buy circle cutters from craft shops. There are also a couple of ideas for folding other shapes knowing about the isosceles triangle.

The mathematics focuses around the properties of the isosceles triangle and ideas of equality (of angles and line segments) and perpendicularity through the lens of symmetry. From these, the tasks connect to properties of other quadrilaterals, measurement, symmetry and angles.


paper folding
congruency
similar triangles
line of reflection
one line of symmetry
angle bisector
perpendicular bisector
properties of the isosceles triangle
of a kite
of a rhombus
equal sides
equal angles
in circles
diameter
cyclic quadrilaterals
radii
chords
angles
right angles
sum of the angles in a triangle
measuring
calculating
parallel lines
alternate angles
complementary
supplementary

A1 Fold an Isosceles Triangle from a Piece of A4 paper. (1)

Task

You need pieces of A4 paper.
Fold an isosceles triangle from one piece of A4 paper.
How many different ways can you do it?

How would you convince someone that the triangle is isosceles?
What different properties of the isosceles triangle do the different folds use?
What is the minimum number of folds needed to get different shapes of isosceles triangles?

Adapting

Measure the angles.
Measure the sides.
Are any of your triangles congruent?
Are any of your triangles similar?

Extending

Use an isosceles triangle that you have folded to fold a kite.

Use a folded isosceles triangle to fold a rhombus.

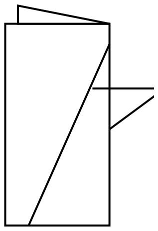
Mathematics

- two sides are the same length
- two angles are the same size
- the isosceles triangle has one line of symmetry
- the angle on a straight line is 180°, the fold creates two equal angles, $180° \div 2 = 90°$
- the line of symmetry divides the triangle into two congruent right-angles triangles
- two sides are equal because of the symmetry when the triangle was constructed, they are mirror images under reflection in the line of symmetry;

- two angles are equal because they match when the triangle is folded along the line of symmetry, they are mirror images under reflection in the line of symmetry;
- a line of symmetry can be a perpendicular bisector and/or an angle bisector

Notes

You only need one fold to get an isosceles right-angled triangle, the fold for creating a square from a rectangle.



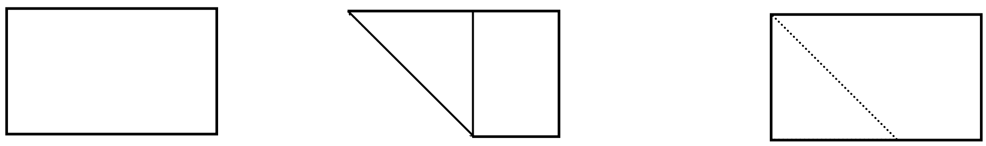
Other isosceles triangles need two folds as in the figure (or fold and straight cut).

Others may require three folds.

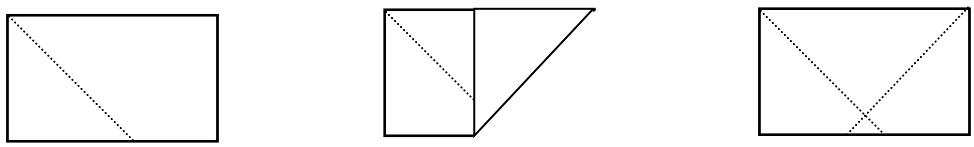
A2 Fold an Isosceles Triangle from a Piece of A4 paper. (2)

Task

You need two pieces of A4 paper.
Fold an isosceles triangle from one piece of A4 paper using the 'make a square' fold.



With the second piece of A4 paper, begin folding in the same way, but then do the 'make a square' fold from the opposite side.



You can now find 3 different isosceles triangles.
How would you convince someone that the triangles are isosceles?

Adapting

What different properties of the isosceles triangle do these folds use?
Are any of your triangles congruent?
Are any similar?
Why?
Can you prove congruence/similarity?

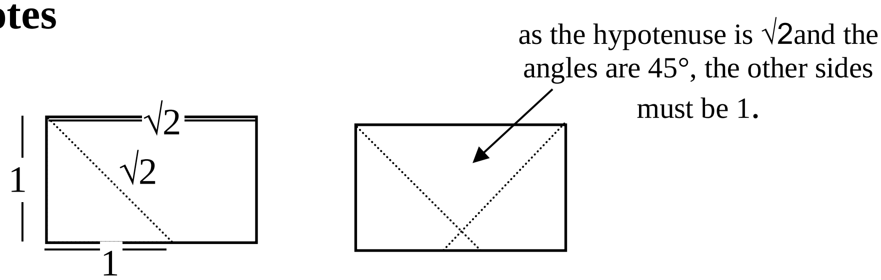
Extending

What happens if you use a rectangle of different proportions?
Try this on squared paper.

Mathematics

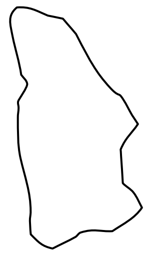
- triangles are similar if they have the same angles
- triangles are congruent when all angles and sides are the same as the corresponding one in the other figure, but it is not necessary to show all of the equalities hence
  - ASA two sides and the included angle
  - SSS all three sides are the same length
  - RHS right angle, hypotenuse and side (special case of ASA)
- the ratio of the sides in A4 paper is 1: $\sqrt{2}$

Notes



A3 Fold an Isosceles Triangle from a Torn Piece of Paper.

Task



Using a piece of paper that has ragged edges, fold an isosceles triangle.
Fold several different triangles from different pieces of paper.

Measure the angles.

How would you convince someone that the triangle is isosceles?

Adapting

What is the minimum number of folds needed to get different shapes of isosceles triangles?

Extending

Use a folded isosceles triangle and fold a kite.

Use a folded isosceles triangle and fold a rhombus.

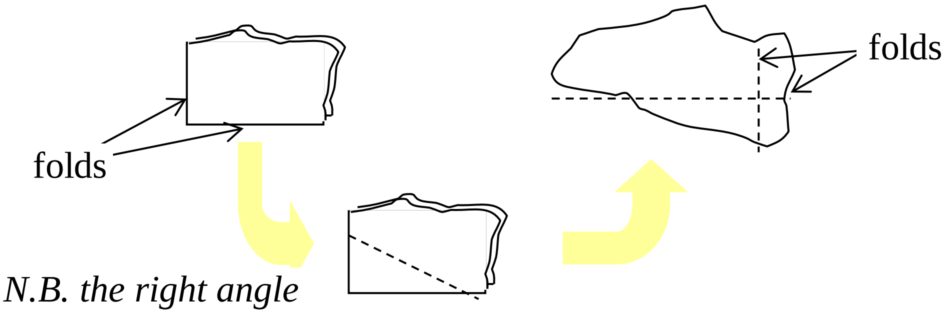
Mathematics

- the lines intersecting at right angles are needed to create the unequal side and the line of symmetry so that the right angles are created, 4 equal angles at a point, $360° \div 4 = 90°$;
- the isosceles triangle has one line of symmetry;
- the line of symmetry divides the triangle into two congruent right-angles triangles;
- two sides are equal because of the symmetry when the triangle was constructed, they are mirror images under reflection in the line of symmetry;

- two angles are equal because they match when the triangle is folded along the line of symmetry, they are mirror images under reflection in the line of symmetry;
- a line of symmetry can be a perpendicular bisector and/or an angle bisector

Notes

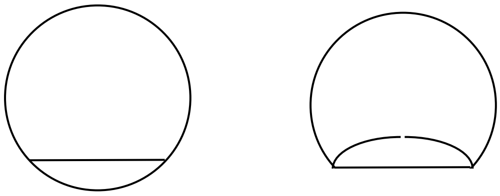
You will need three folds.



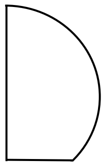
A4 Folding an Isosceles Triangle using Filter Paper.

Task:

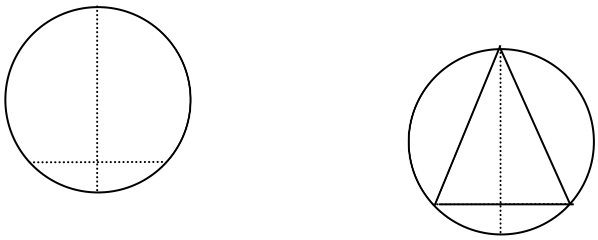
Using a circular filter paper, fold along a chord



Fold the shape along its line of symmetry.



Open up the folds.



Join the ends of the chord to the point on the major arc.

Why is the shape an isosceles triangle?

Adapting

Join the ends of the chord to the point on the minor arc to make a kite.
How do you know the shape is a kite?

Extending

Try this with your second fold in different positions.
How do the angles change?

Mathematics

- the two folds produce a right angle
- properties of the isosceles triangle; perpendicular bisector as the line of symmetry
- the line of symmetry of a circle passes through its centre
- chord and line of symmetry as a part of the construction of an isosceles triangle
- one pair of angles in the kite will always be 90º (angle in a semi-circle)
- the line of symmetry of the chord is a diameter of the circle
- chord and radius though centre of chord are perpendicular to each other

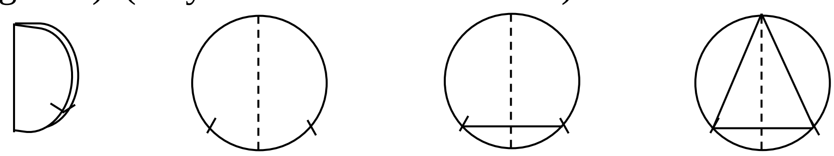
Notes

If you use this task to provide practice of measuring angles, be prepared for the inaccuracy and that the angles will not necessarily appear to sum to 180°.
How might you work on the number of angles that it is necessary to measure?
One of the circle theorems states that the perpendicular bisector of a chord passes through the centre of the circle.
In what other ways can you link this task to the beginnings of the geometry of circle theorems?

A5 Using Filter Paper 2

Task

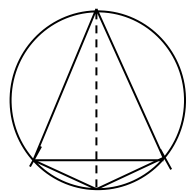
Take a circle of filter paper and fold a line of symmetry [N.B. be very careful that the two halves match up.] Open up the circle – why is the line of symmetry a diameter of the circle? How would you convince someone else?
Refold the circle (use the same line of symmetry). Mark a point on the circumference and its corresponding symmetry point (figure a).
Open up the circle and join the two points with a straight line; a chord, (figure b, c).
Join the end points of the chord to the point on the circumference that is on the line of symmetry – the one that is furthest away (figure d). (Or you could fold the chord.)



a b c d

Measure lengths and angles. Which lengths and angles should be the same size? Why?

Now join the ends of the chord to the other end of the line of symmetry.



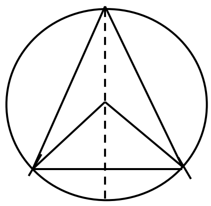
Measure the angles and lengths.
Are there any connections with the angles in the first triangle? Describe these.

Adapting

Get another piece of filter paper and use the line of symmetry to produce a chord (longer or shorter than last time).
Produce the triangles.
What sort of triangles do you get? Why?
What other shapes can you identify?

Extending

Begin the activity as before. Once you have the triangle, find the centre of the circle and join the ends of the chord to the centre of the circle.



Which angles we the same/different

Mathematics

- line of symmetry of a circle passes through the centre of the circle
- perpendicular bisector of a chord (the line of symmetry) passes through the centre of the circle
- the line of symmetry is a mirror line with the sides of the isosceles triangle being mapped onto each other under reflection, proof of equal length
- measuring lengths and angles
- properties of an isosceles triangle
- properties of the kite, line of symmetry, one pair of opposite angles are equal, the other pair sum to 180°
- properties of cyclic quadrilaterals

Other notes

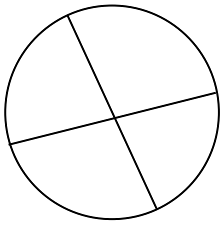
The kite produced this way is cyclic. Are all kites cyclic?
A cyclic quadrilateral has all its vertices on the circumference of a circle.

The angle in a semicircle is 90º.

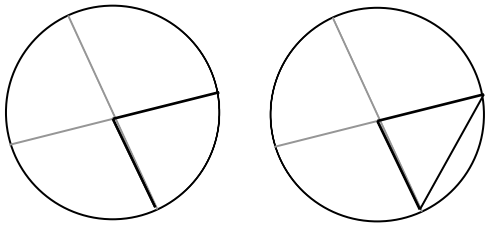
A6 Using Filter Paper 3

Task

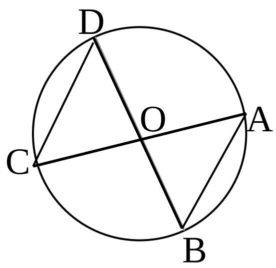
Take a circle of filter paper and fold a line of symmetry [N.B. be very careful that the two halves match up.]
Open up the circle – why is the line of symmetry you have just folded a diameter of the circle? How would you convince someone else?
Fold another line of symmetry (try not to do it at right angles to the original one.)



Draw (on the lines of symmetry) two adjacent radii, do this as accurately as possible.
Draw a chord joining the two radii.
What sort of triangle have you drawn?
How would you convince someone of this?



Draw the opposite triangle.
Labelling the votives is useful when talking about the angles.
Measure the angles of the triangle.
Which angles are the same size? Why?

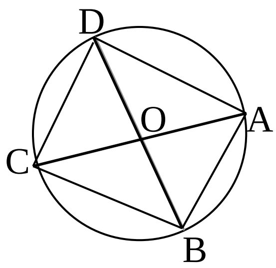


Are any lines in the diagram parallel?
Does the drawing have any lines of symmetry?

Construct some different triangles in the same way.

Adapting

Draw the other two chords.



What shape is ABCD? Why?
Are the lines of symmetry the same?

Why are the original folds (the lines of symmetry of the circle) not lines of symmetry of the drawing?

Extending

Try this with other sized circles.
Do you ever get similar shapes? How?

Mathematics

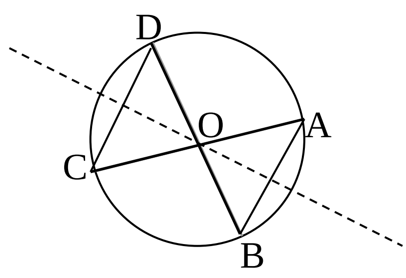
- properties/language of circle
- properties of isosceles triangle
- symmetry and congruence
- parallel lines
- vertically opposite and alternate angles.

Other notes

Language – use rather than expect pupils to learn:

semi-circle
diameter
centre
radius
chord
sector/segment – major, minor

The diagram has two lines of symmetry, with the line shown also being the line of symmetry for the two isosceles triangles AOB and COD



$\angle$OAB is equal to $\angle$OBA (angles in an isosceles triangle where OA and OB are radii of the circle.)

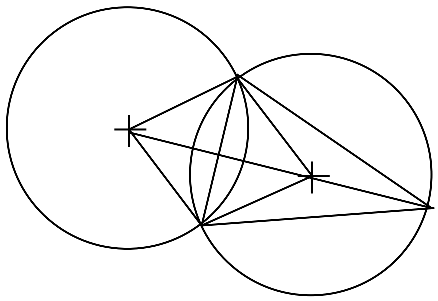
CD is parallel to BA (given the other line of symmetry and alternate angles.)
$\angle$AOB = $\angle$COD (vertically opposite/symmetry.)

A7 Using 2 Circles of Filter Paper.

Task

Take two circle of filter paper the same size. Mark their centres. Overlap them and stick them together. Mark the points of intersection of the overlap.
Draw the radii to the points of intersection and the common chord.
Join the centres of the circles and extend this line to draw a diameter on the top circle.
Draw the chords from the points of intersection to the end of this diameter.
Describes the shapes you have (lengths, angles, symmetry.)
Explain your results.

Repeat the activity; will you ever get a square? When?
Are any right angles possible?



Adapting

Try the activity with 2 different sizes of filter paper.
Do you ever get right angles? When?
Can you ever get a square?

Extending

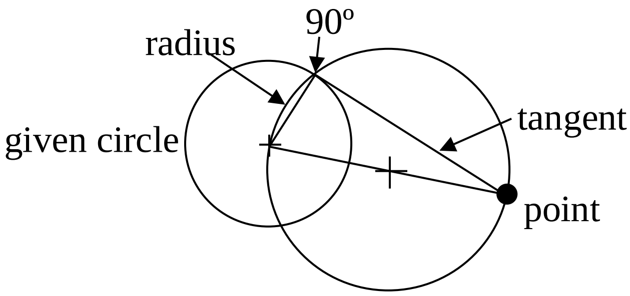
Lines which touch the circumference of a circle and are at right angles to the radius at that point are called tangents.
When do you get tangents?

Mathematics

- properties of isosceles triangle, rhombus, kite, square
- angle in a semi-circle is a right angle
- symmetry
- towards the construction of tangents to a circle from a given point

Notes

To construct a tangent to a circle from a given point, you need the semi-circle with the diameter being from the point to the centre of the given circle. The intersection point gives you the point on the circle.

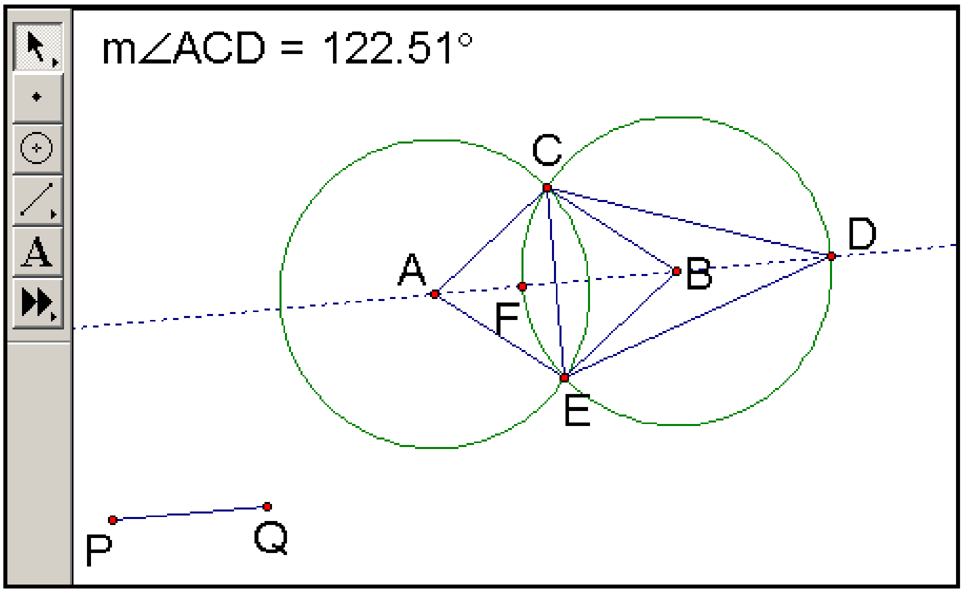


A DGS version of 'Overlapping Circles' is over the page.

Overlapping Circles (DGS) More of A7

Task

Construct two identical circles. It can help to have a line segment that you can use as a vector (PQ in the screen dump) for the translation of the centre point.
Let them overlap.
Join the centres with a line.
And join the radii from centres to points of intersection, the common chord and the end of the diameter on one circle to the points of intersection.
Where are the isosceles triangles?
What other polygons are there?



Is angle ACD ever 90º? When?
Change the size of the circles by dragging Q.

Adapting

Try the construction with two different sized circles. (Use two vectors.)

Extending

Draw a circle and a point.
Construct a tangent from the point to the circle.
Construct a second tangent.
Can you find any isosceles triangles?

Mathematics

- translation
- circle language
- properties of shape
- angles in a semi-circle
- towards constructing tangents from a given point

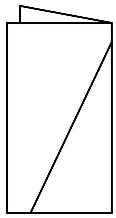
Other notes

DC is the tangent to the circle centre A from the point D.

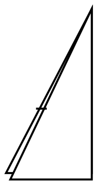
A8 From ANY Rectangle.

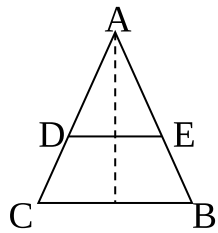
Task

Fold a rectangle of paper very carefully along its line of symmetry.
With the paper folded draw a line from a point on the fold to an edge at the end of the fold.
With the paper folded cut accurately along the line.
Open up the fold.
What sort of triangle have you made? Why?

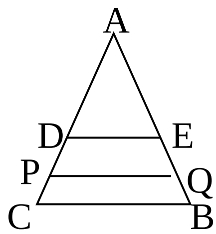


Refold the triangle (the fold is the line of symmetry).
Mark a point on the cut edge of the triangle, so that the mark can be seen on both sides.
Open up the triangle and join the points.





What sort of triangle is ADE? Why?
Which angles are the same size? Why?
Are any lines parallel? Why?



Try marking another pair of points.
Mark equal angles and any parallel lines.

Adapting

Cut out a different isosceles triangle using the same rules and repeat the above.

Extending

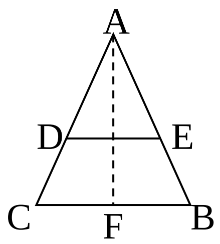
Can you find similar triangles?

Use two folds and the two isosceles triangles forming a rhombus.
Make an identical rhombus with another piece of paper.
Get sets of parallel lines in two different ways.

Mathematics

- properties of isosceles triangle
- similarity
- parallel lines
- corresponding angles
- external angles of a triangle
- properties of isosceles trapezia

Notes



The triangle cut out should be isosceles as it has one line of symmetry.
There are many different ways of describing the geometry of the shape using the symmetry of the construction.
For example:

- AD = AE (symmetry.)
- C and B and D and E are image pairs under reflection in the line of symmetry.
- The perpendicular distance from C to AF is the same as the perpendicular distance from B to AF. (⊥ distance D to AF = ⊥ distance E to AF.)
- ΔADE is similar to ΔABC (AAA, all angles are the same.)
- DE is parallel to CB (corresponding/supplementary angles.)

A9 Triangles from Rectangles

Task

Take a rectangular piece of paper and draw the diagonals. Cut carefully along them
Describe the four triangles as carefully as you can.

Each of the triangles has one line of symmetry. Make folds to show these.
What shapes can you make with two or more pieces that have exactly one line of symmetry?
What shapes can you make with two or more pieces that have exactly two lines of symmetry? (It may help to draw or fold the lines of symmetry.)

Adapting

Try using rectangles of different sizes.
How do the angles in the triangles change?
When do you get similar triangles? (In the same rectangle, between different rectangles.)

Extending

What fraction of the area of the rectangle is the area of each triangle? Why?
What is the relationship between the areas of the 2 different triangles?
Connect the work to trigonometry and the tangent of the ratio of the sides.

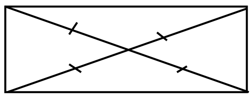
Mathematics

- the four triangles are isosceles
- the diagonals of a rectangle bisect each other – equal lengths as sides of the triangles
- symmetry
- bisection of angles and sides
- diagonals of a rectangle are not lines of symmetry of the rectangle (N.B. square is a special case)

Notes

Each pair of congruent triangles will make a rhombus – with lines of symmetry which bisect at right angles.

The equal sides of the triangles are the same size – this may be 'obvious' but it is not something that is usually made explicit. The symmetry activity is intended to stress this.

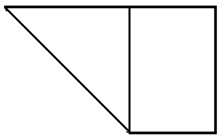


The area of each triangle is the same – this is counter-intuitive and is worth working on.
Similar triangles will be cut from rectangles with sides in the save ratio, e.g. 4 cm by 8cm with 6cm by 12cm.

A10 Fold a Square

Task

Take an A4 piece of a paper, make the diagonal fold to make a square.
Cut off the piece that is not needed.
Open up the square.
How many isosceles triangles can you see?
Describe them in detail.

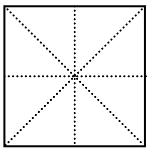


Fold the other diagonal.
How many isosceles triangles are there?
Describe them in detail.

Now fold the other two lines of symmetry of the square.
How many isosceles triangles are there now?
Describe them.
What fraction of the square is each type of isosceles triangle?
What fraction of 360° is each of their odd angles?

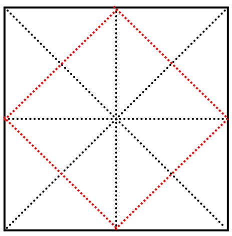
Adapting

Cut the square into the 8 small right angled isosceles triangles and use the pieces to make other shapes – describing their angles, lengths (in terms of the sides of the original square), their areas (in terms of the original square) and their symmetry.



Extending

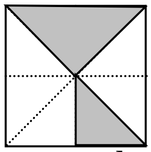
Fold a new square and add other folds – the diagonal of the 4 small squares.
Use these pieces to explore fraction of length and area.



Mathematics

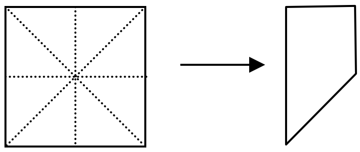
- properties of square – diagonals bisect at right angles, lengths same, angles 90º
- angle bisection
- similarity
- area
- fractions
- symmetry

Other notes



Cut shapes make it easy to compare lengths and angles but can be inaccurate.

Explore what plane shapes you might make if the shapes are folded but not cut.



Section B: Isosceles triangles in Circles

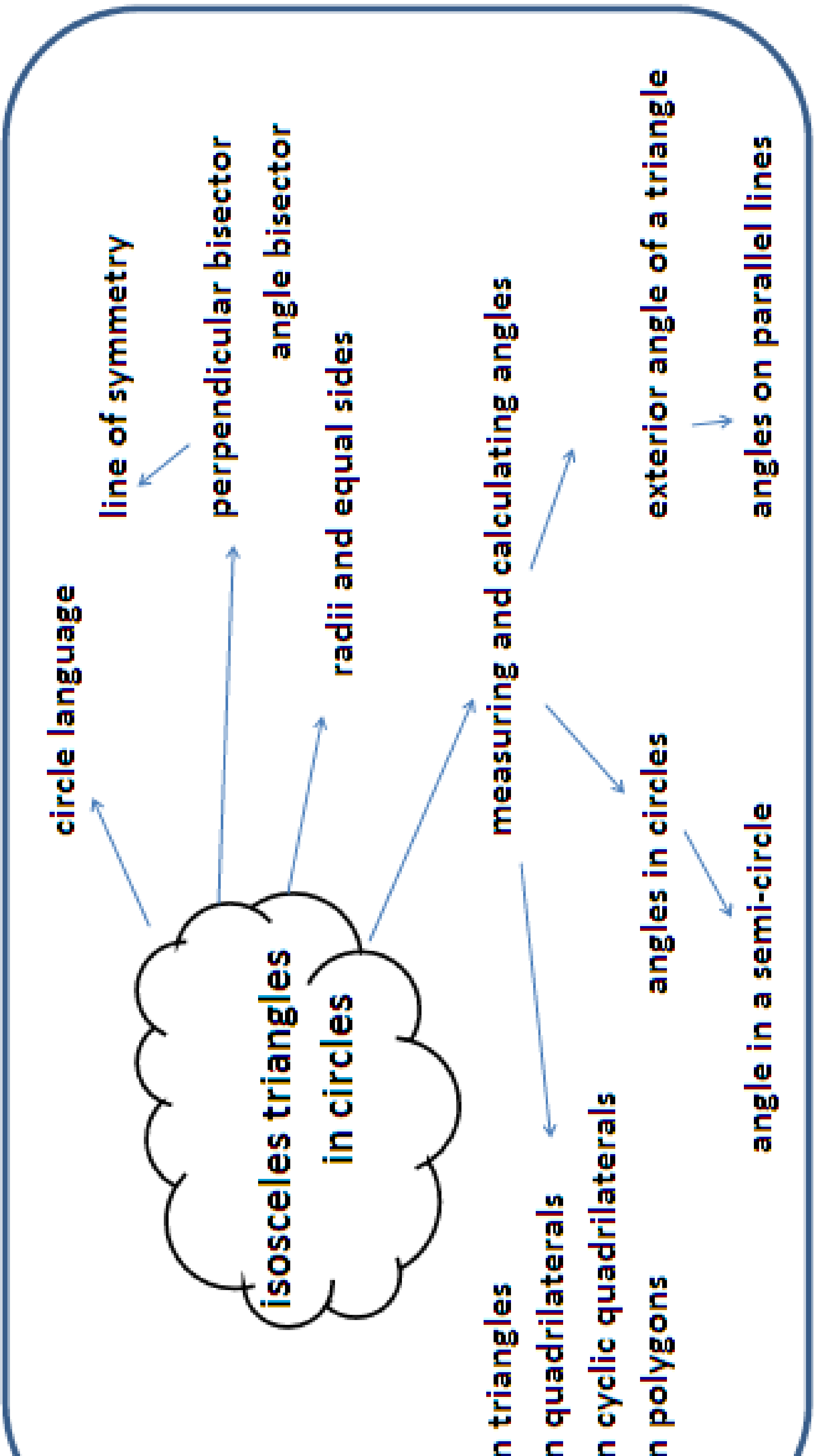
Section notes

In this section there are various activities which connect the circle to the isosceles triangle. Two radii become the two equal sides with a chord as an odd side. Using the perpendicular bisector of the triangle extended through the centre of the circle lots of geometry and connections follow.

You can play with these tasks at any level if the students are working on or know the properties of the isosceles triangle. The level of formalisation of circle geometry which connects remains your choice for the class.

Minimally if the student knows about the isosceles triangle much geometry can be deduced. There is no need to over load the memory!

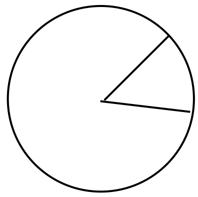
A starting point for many of the tasks in this section is to actually construct a circle. You can decide when the class is ready for a sketch that might represent a generality of the problem. Most students in our experience will need construct an actual circle and then make the measurements of length and angle. Since measurement is inaccurate generality is not always easy to ‘see’.


isosceles triangles in circles
circle language
perpendicular bisector
line of symmetry
angle bisector
radii and equal sides
measuring and calculating angles
exterior angle of a triangle
angles on parallel lines
angles in circles
angle in a semi-circle
in triangles
in quadrilaterals
in cyclic quadrilaterals
in polygons

B1 The Isosceles Triangle in the Circle.

Task

Using a pair of compasses, draw a circle of radius 8cm.
Draw a radius.
Draw another radius so that the angle between the two line segments is acute.
Join the ends of the radii (a chord) to form a triangle.



- Why is the triangle isosceles?

Measure the angles of the triangle.
Compare results with other class members.

Adapting

Draw the circle and one radius – measure an angle of 40° at the centre of the circle and draw another radius to form the angle.
Complete the triangle.
What are the other angles? Why?

Repeat the task with angles of 80°, 120°, 100°, 50°, 60°, 150°.
Predict the other angles before you measure them

Extending

Find the perimeters of each triangle.
Construct the triangles on squared paper – make sure the centre of the circle is at an intersection of squares. Use the squares to find the area of the triangles.

Mathematics

- properties of the circle and the isosceles triangle; equality of radii and the equal sides of an isosceles triangle
- connecting radii with equality and the sides of the triangle
- measuring and calculating angles.

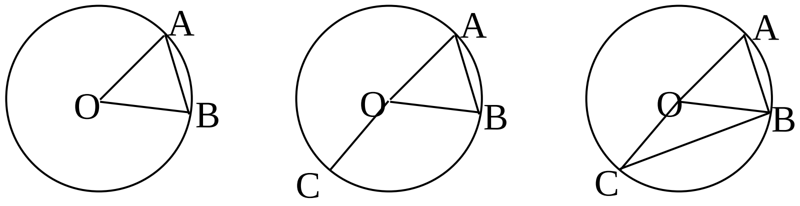
Other notes

The exact relationships will be subject to errors from construction and measurement.
It is difficult for some children to see the radii as equal lengths even if they “know” the fact. The task is designed to make the connections as explicit as possible.

B2 Two Isosceles Triangles on the Diameter.

Task:

Using a pair of compasses, draw a circle of radius 8cm.
Draw a radius.
Draw another radius so that the angle between the two line segments is 50°.
Join the ends of the radii (a chord) to form a triangle.
Extend one of the radii to form a diameter.
Draw another chord to create another triangle.



Measure the angles of both of the triangles.
What is the size of angle ABC?
How is angle BOC related to angle BAO?
How is angle AOB related to angle ACB?

Compare results with other class members.

Adapting

Repeat the task with an angle of 40°.

Repeat the task with angles of 80°, 120°, 100°, 40°, 60°, 150°.
Predict the other angles before you measure them.

Extending

For each triangle measure AB and BC.
Square the lengths and find the sum of the two squares.
What happens?

Repeat the diagram for another angle – say 48°, does the relationship still work?
Try the diagram with a 10cm circle or a 6cm circle.

Mathematics

- properties of the circle and the isosceles triangle; equality of radii and the equal sides of an isosceles triangle
- measuring and calculating angles.
- working towards angle in the semi-circle is a right-angle
- working towards the exterior angle of a triangle is equal to the sum of the two opposite interior angles.

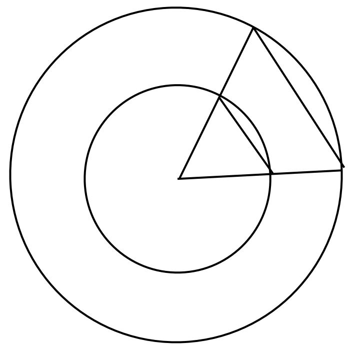
Other notes

The exact relationships will be subject to errors from construction and measurement.
Reinforce the angle on a straight line is 180°.
The discussion can include working towards generality, possibly some aspects of the circle theorems:

- The angle at ABC is a right angle – the angle subtended by the diameter.
- The angle ABC is half the angle at the centre of the circle (180°).

B3 Nesting Isosceles Triangles in Circles.

Task:



Using a pair of compasses, draw a circle of radius 10cm.
Draw a radius.
Draw another radius so that the angle between the two line segments is acute.
Join the ends of the radii (a chord) to form a triangle.

- Why is the triangle isosceles?

Measure the angles of the triangle.

Now draw a circle of radius 5cmwith the same centre as the first circles.
Draw the chord

Measure all the angles.
Measure the lengths of the chords.

Adapting

Draw a 10cm circle – measure an angle of 40° between the two radii. Complete the triangle drawing the chord.
Draw a circle of radius 5cm with the same centre (concentric circles)
What are all the other angles? Why?
Meaure the chords. How are they related?
Repeat the task with angles of 80°, 120°, 100°, 50°, 60°, 150°.
Predict the other angles before you measure them

Extending

Collect the measurements of the angle at the centre of the circle and the lengths of the chords.

Predict the lengths of the chords if the centre angle is 70° - draw the diagram to check.
Try to predict the lengths of chords for other angles – check by drawing.
Draw a graph (you could use a spreadsheet/graphical package) for each of the ordered pairs:

(angle, larger chord length)
(angle, smaller chord length)

Mathematics

- properties of the circle and the isosceles triangle; equality of radii and the equal sides of an isosceles triangle
- measuring and calculating angles
- corresponding angles and parallel lines
- similar triangles
- circle language: radius, chord, concentric.

Other notes

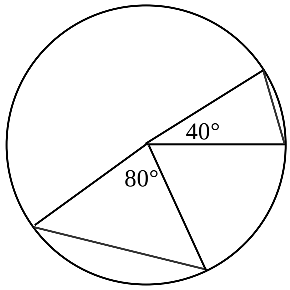
The triangles are similar so the chord lengths are in the ration 1:2.
The chords are parallel because of the angle values.
The chord lengths are 10sin(angle/2) and 5sin(angle/2), the graph will give the first part of the sine wave.

B4 The Isosceles Triangle and Chord Length.

Task:

Draw a circle of radius 8cm and one radius.
Measure an angle of 40° at the centre of the circle and draw another radius to form the angle.
Draw the chord to form the isosceles triangle.
Measure the length of the chord.

In the same circle construct another isosceles triangle with an angle at the centre of 80°.



Is the chord twice the size of the chord in the 40°?
If not, what angle would you need for the chord to be twice the size?

Adapting

Try the task for different angles.
For what angles does this task become impossible.

What construction can you use to help with this task?

Extending

Draw an isosceles triangle inside the circle, with the angle at the centre of 40° as above.

Draw another isosceles triangle inside the same circle so that the perimeter of the new triangle is twice the perimeter of the original.

Mathematics

- measurement of lengths
- chord length is not directly related to angle size (an introduction to sine?)
- circle language: radius, chord.

Other notes

In an 8cm circle

Angle at centre	Half-angle	Chord length (1d.p.)	New angle (nearest degree)
30	15	4.1	62
40	20	5.5	86
50	25	6.8	115
60	30	8.0	180

Construction: Construct an arc of twice the length of the measured chord by opening a pair of compasses to the distance and marking off a point on the circle.

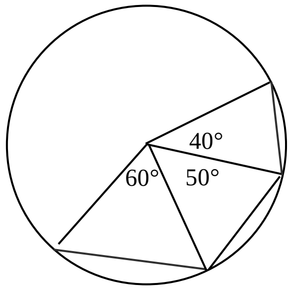
B5 Three Isosceles Triangles in a Circle.

Task:

Draw a circle of radius 8cm and one radius.
Measure an angle of 40° at the centre of the circle and draw another radius to form the angle.
Draw the chord to form the isosceles triangle.
At the centre draw an angle of 50°, draw the radius. Then an angle of 60° and the chord.

The three isosceles triangles form a pentagon. How?

Find the angles in the pentagon. (Check by measuring.)
Find the sum of the angles.



Adapting

Try the task for different angles.
When do you draw a pentagon with one line of symmetry.
Which angles are the same size?

Extending

When do the three triangles form a quadrilateral? (See Task B6.)
Can the shape ever be a trapezium, kite, parallelogram?

Mathematics

- calculating angles
- angle sum in a pentagon
- circle language: radius, chord.

Other notes

The sum of the interior angles of a pentagon is 720°
The pentagon in the diagram is not cyclic as are vertex is at the centre of the circle rather than being on the circumference.

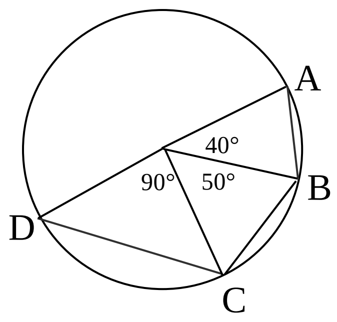
B6 Three Isosceles Triangles on a Diameter.

Task:

Draw a circle of radius 8cm and one radius.
Measure an angle of 40° at the centre of the circle and draw another radius to form the angle.
Draw the chord to form the isosceles triangle.
At the centre draw an angle of 50°, draw the radius. Then an angle of 90° and the chord.

The three isosceles triangles form a quadrilateral. How?

How would you convince others?



(a) Find the angles in the quadrilateral. (Check by measuring.)
(b) Find the sum of the angles.
(c) Find the sum of angles DAC and BCD (opposite angles).
(d) Find the sum of angles ABD and CDA (opposite angles).

Make and cut out two copies of the quadrilateral ABCD – how can you use these to show the results of parts c and d. Use four copies to show the result of b.

Adapting

Try the task for different sets of angles that sum to 90.
When do you draw a quadrilateral with one line of symmetry.
Which angles are the same size?

Extending

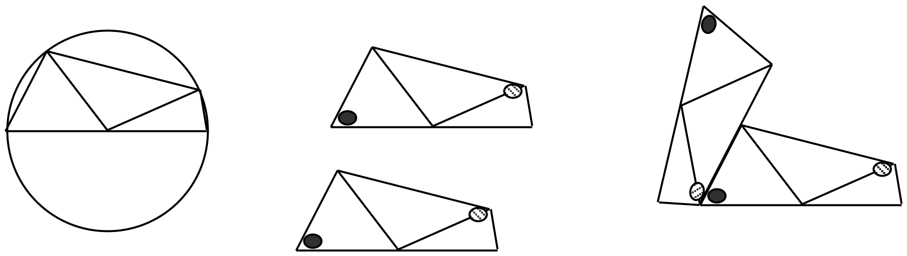
Using the diameter as a mirror, reflect the 3 triangles to form a hexagon. Find the angle sum of the interior angles.

Mathematics

- calculating angles
- angle sum in a quadrilateral
- opposite angles in a cyclic quadrilateral
- circle language: radius, chord
- justification

Other notes

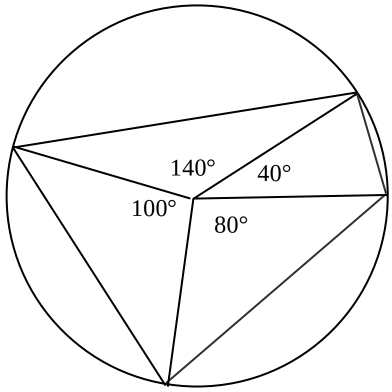
The sum of the interior angles of a quadrilateral is 360°. Opposite angles of a cyclic quadrilateral are supplementary (sum to 180°).



B7 Four Isosceles Triangles and the Cyclic Quadrilateral.

Task:

Draw a circle of radius 8cm.
Draw 4 radii so that the angles between them are 40°, 80°, 100°, 140°.



Find the angles in each of the isosceles triangles.
Use the results to calculate the angles in the quadrilateral.
Repeat for different angles at the centre.

What angles at the centre do you need for the quadrilateral to be a square?

Adapting

What angles at the centre do you need for the quadrilateral to be a rectangle?

By drawing other chords (use dotted lines) what other isosceles triangles can you find?
Find their angles.

Extending

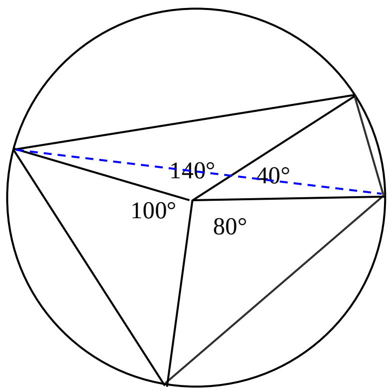
Prove that for any cyclic quadrilateral opposite angles add up to 180°.
Prove that any quadrilateral with opposite angles that add up to 180° is cyclic.

Mathematics

- angles in an isosceles triangle
- angles in a triangle
- angles in a quadrilateral
- angle relationships in a cyclic quadrilateral

Other notes

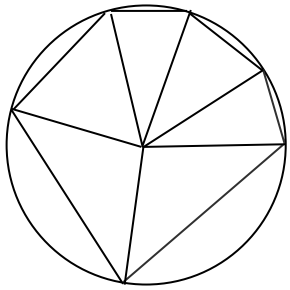
Opposite angle of a cyclic quadrilateral are supplementary, they add up to 180°
The diagram given has chords which are diameters but it does not look like it on the diagram which is not accurate.



B8 Six Isosceles Triangles in a Circle.

Task:

Draw a circle of radius 8cm.
Draw 6 radii. (Have the radii spread round the circle – avoid reflex angles – explore them later.)
Draw the chords that join adjacent radii.
What polygon is formed from the chords?



Find the angles in each of the isosceles triangles.
Use the results to calculate the angles in the polygon
Repeat for different radii.

What relationships between the angles at the centre do you need for the polygon to have exactly one line of symmetry?

Adapting

What angles at the centre do you need for the polygon to have exactly two lines of symmetry?
What angles at the centre do you need for the polygon to have exactly three lines of symmetry?

What angles at the centre do you need for the polygon to have exactly four, five, six lines of symmetry?

Extending

What happens if one of the angles between the radii is reflex?

Generalise the rule for one line of symmetry to other cyclic polygons with an even number of sides.
How many lines of symmetry are possible in a cyclic polygon?

Mathematics

- angles in an isosceles triangle
- angles in a triangle
- angles in a polygon
- angle relationships and symmetry

Other notes

We often begin the construction of regular polygons without any play with the ideas, so that the use of isosceles triangles is told rather than pupils knowing why they are necessary.

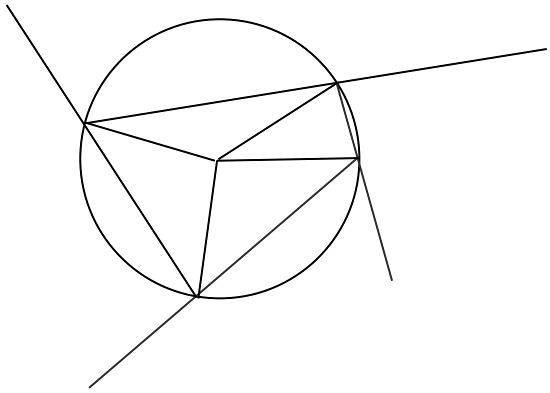
The formula for the internal angles of a hexagon could be worked on as:

sum of angles in a hexagon
= angles of 6 triangles – centre angle

B9 External Angles

Task

Draw a circle of radius 8 to 10 cm.
Draw 4 radii. (Avoid reflex angles.)
Draw the chords between adjacent radii.
Measure the angles in the shape.



Extend the sides of the quadrilateral in one direction,
so that the spokes follow the same way round.
Measure or calculate these new angles?

These are called the external angles.
Find their sum

Do this for another circle and quadrilateral.

Adapting

Extend the sides of the quadrilateral in the opposite direction.
Begin the task with 5 radii, or 6 or 8 or …

Extending

Prove that the sum of the external angles of a cyclic quadrilateral has to be …
Prove that the sum of the external angles of a cyclic polygon has to be …

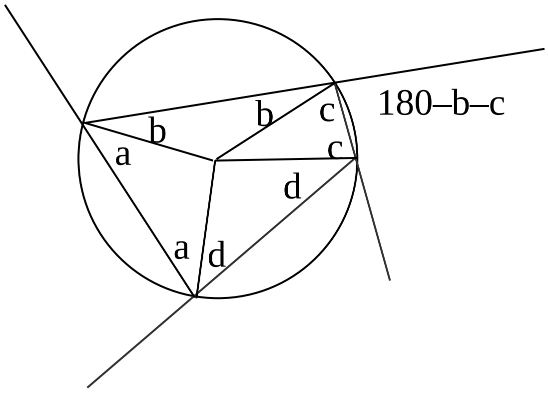
Mathematics

- Angles in isosceles triangles
- Angles on a straight line 180
- Angles at the centre of the circle
- Justifying
- Proof

Other notes

The isosceles triangles allow you to connect the angles at the centre to the external angles.

centre	external
180-2a	180-a-b
180-2b	180-b-c
180-2c	180-c-d
180-2d	180-d-a
720-2a-2b-2c-2d	720-2a-2b-2c-2d
360	360



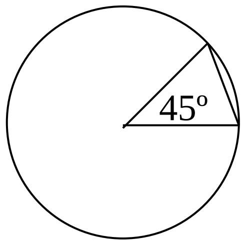
This does not mean you that you ignore the simpler idea of walking around the shape – doing a full turn.

Reflex angles cause a complication which can act as an extension.

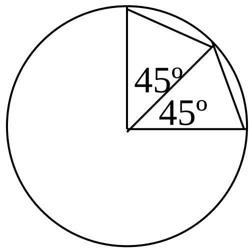
B10 Equal Angles at the Centre

Task

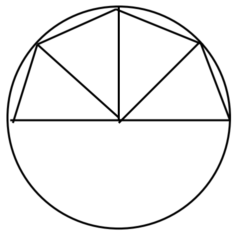
Draw a circle of radius 8cm.
Draw a radius.
Draw an angle of 45º at the centre.
Construct the triangle using the two radii and the chord.
Measure (calculate) the angles. Measure the sides.



Construct another 45º angle and hence another triangle.
Measure this one.
Is it congruent to the first triangle? Why?



Keep constructing more triangles the same way.
How many will fit?
How would you check for error?



Adapting

Try this with a 60º angle.
Try this with a 40º angle.
Try this with a 120º angle.

Extending

What angles will use all the space at the centre so that you obtain a regular polygon?
Test your ideas using a DGS.

Mathematics

- measuring, drawing angles
- measuring lengths
- properties for congruence of triangles
- equal angles at the centre give equal chords
- equal chords subtend equal angles at the centre of the circle
- angle at centre is 360º
- factors of 360
- error on measurement versus calculation

Other notes

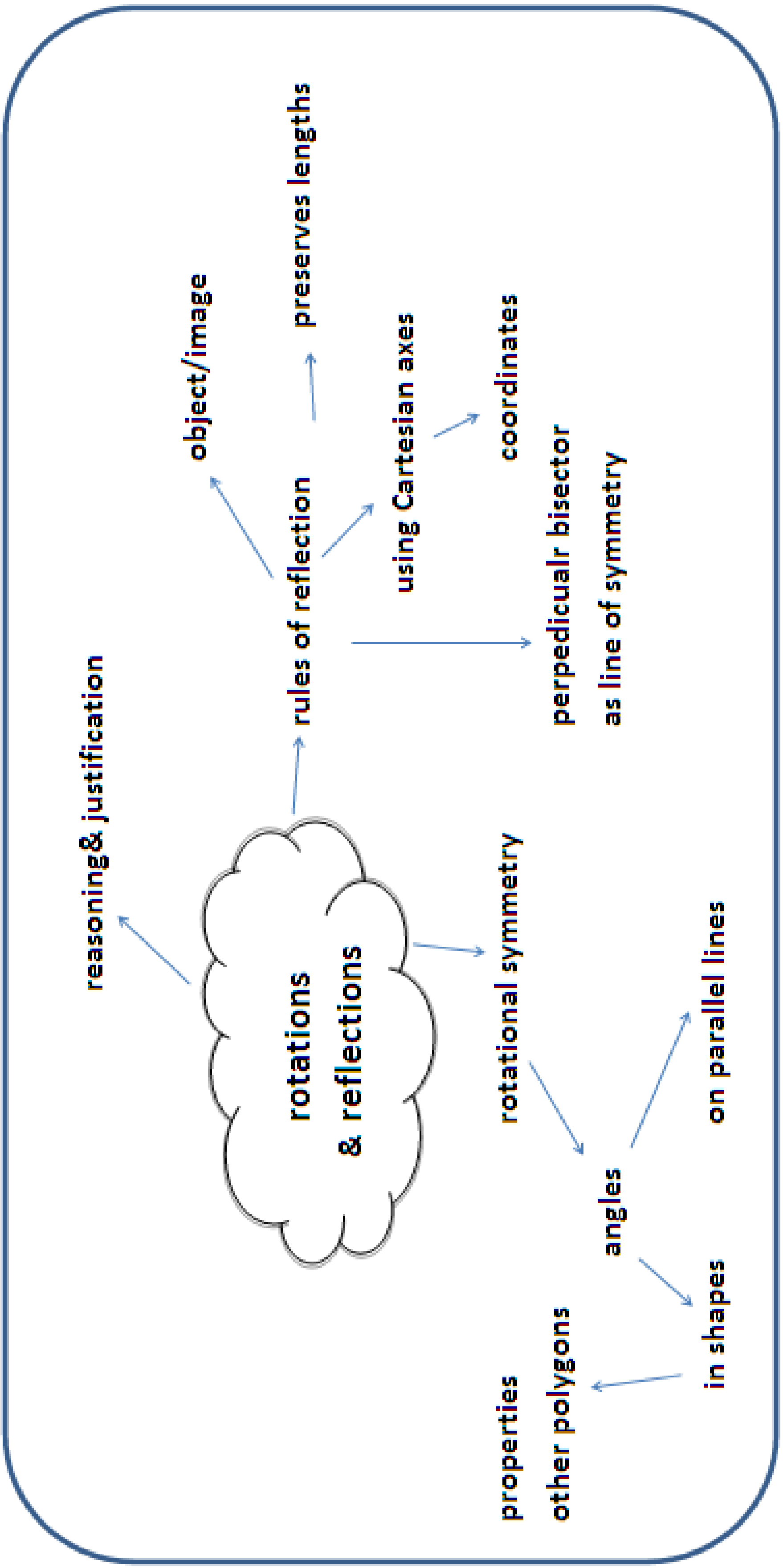
Repeated construction is likely to lead to cumulative errors – how will you persuade pupils of the difference between the idealist and realistic construction.

Section C: Grids and Axes

Section notes

The tasks in this section relate the properties of the isosceles triangles to properties of the rectangle (and hence square) and explore the different properties of these shapes... and more besides! Similar triangles can become part of the explorations because of all those parallel lines, as can various properties of angles. There are lots of links – the kite, rhombus, Pythagoras, some of the extensions ask for algebraic proof.

Squared paper can be used for an early play with these relationships and connections. Perhaps with older pupils coordinate grids and axes and equations of lines might be sketched. The lack of accuracy in sketches makes pupils stretch their knowledge of the maths.



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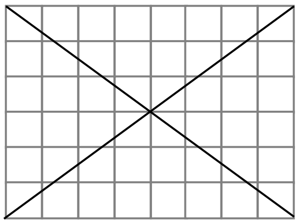
C1 Rectangles on Squared Paper.

Task

On centimetre squared paper draw a 6cm by 8cm rectangle.
Draw the diagonals. How long are they?



How many isosceles triangles have you got?
How would you convince someone that the triangle is isosceles?

Measure the angles.
Measure the sides.
Find the area of each triangle and compare this to the area of the rectangle.
Explain your findings.

Adapting

Use different sizes of rectangles.

Extending

Make a table that shows the angles of the triangles in the different rectangles and the ratio of the sides of the rectangle.
How do the angles and ratios relate?

Draw a circle centre where the diagonals cross that goes through the vertices of the rectangle.

Mathematics

- the diagonals of a rectangle bisect each other
- the area of each triangle is one quarter of the area of the rectangle
- the equal sides of the different triangles are equal to each other
- justification
- explanation based on geometrical properties

Other notes

The equality of the sides of the triangles is ‘obvious’ but rarely made explicit – it is one of the reasons why a rectangle is a cyclic quadrilateral.
The area of the triangles is counter-intuitive and needs to be worked on – can it really still be true in a 2 by 20 rectangle?

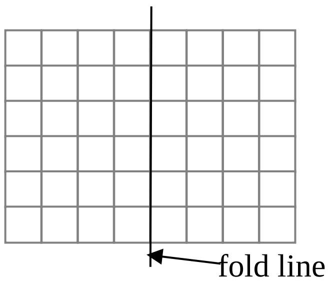
Encourage arguments based on symmetry

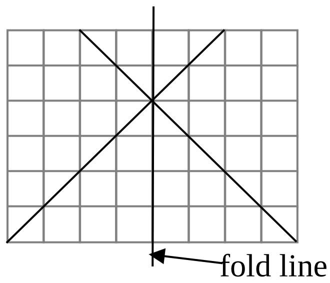
- the diagonals an at the centre of rotational symmetry, where the lines of symmetry cross
- imagine the lines of symmetry, how does this inform equality of length and angle?

C2 Using Rectangles of Squared Paper.

Task

Cut a 6 by 8 rectangle from centimetre squared paper.
Fold the shorter line of symmetry.
Plot a point on the fold line which is more than 3 squares from the bottom.
Join the bottom corners to this point and extend to the other side.





How many isosceles triangles have you got?
How would you convince someone that the triangle is isosceles?

Are the triangles similar?
How do you know?

Measure the angles.
Measure the sides.
What is the ratio of corresponding sides?
How might you work out all the angles having measured one?

Adapting

Use different sizes of rectangles.
How do the angles and ratios change?
How could you begin to predict this?

Extending

What are the areas of the different triangles?
How does the ratio of their area relate to the ratio of their sides?

Mathematics

- the line of symmetry of an isosceles triangle divides the triangle into two congruent right angled triangles
- triangles are similar if they have the same angles
- in similar triangles corresponding sides are in the same ratio
- the ‘odd’ sides of the isosceles triangles are parallel – so you have corresponding and alternate angles
- justification
- explanation based on geometrical properties

Other notes

Recognising the isosceles triangles can be difficult for some pupils, so work on ways to help then recognise the combination of two congruent right angled triangles.
Stress the role of symmetry.

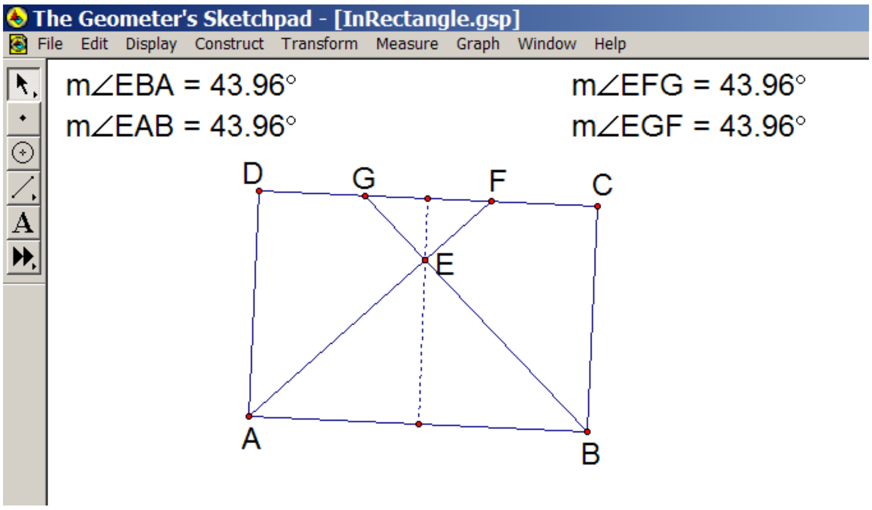
C3 Rectangle and Line of Symmetry using a DGS

Task

Construct a rectangle ABCD, (in the picture AB can be made longer and C controls the other pair of sides) by constructing:

- line segment AB
- a perpendicular to AB at B
- a point on the perpendicular, C
- a perpendicular to BC at C
- a perpendicular to AB at A.
- Construct the intersection.
- Use line segments and hide construction lines.

Then construct a line of symmetry (the shorter one).
Construct a point on the line of symmetry, E.
Construct a ray from A through E, another from B through E.
Mark the intersections on CD (F, G).
Replace rays with line segments.



Are the illustrated angles always the same?
What is the range of angles possible, if E says inside ABCD and there are 2 triangles?

Adapting

Do the construction on a square grid.
Find the ratio of the sides of the triangles for different sizes of rectangles.

Extending

Find the area of the triangles.
How does the ratio of their areas relate to the ratio of their sides?

Mathematics

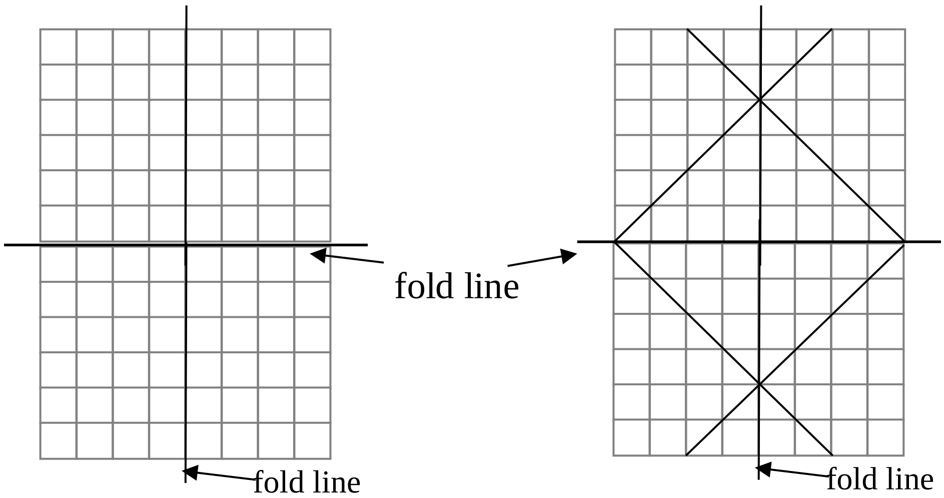
- the diagonals of a rectangle bisect each other
- the area of each triangle is one quarter of the area of the rectangle
- the equal sides of the different triangles are equal to each other.

Other notes

This is included in this section to show an alternative context for C2.

C4 Two folds

Cut a 12 by 8 rectangle from centimetre squared paper.
Fold the shorter and the longer line of symmetry.
Plot a point on the vertical fold line which is more than 3 squares from the horizontal fold line. Join the corners of the horizontal fold line to this point and extend to the other side. Reflect this shape in the horizontal fold line.



How many isosceles triangles have you got? What is the name of the quadrilateral formed?
How would you convince someone that the triangles are isosceles? How does this help to name the quadrilateral?

Which triangles are similar? How do you know?
How might you work out all the angles having measured one?
Measure the sides.
What is the ratio of corresponding sides? When does the rhombus become a square and why?

Adapting

Use a different first point
Use different sizes of rectangles.
How do the angles and ratios change?
How could you begin to predict this?

Extending

Find the areas of the different quadrilaterals?
How does the ratio of their area relate to the ratio of their sides?
Instead of reflecting the first triangle created to make a rhombus, choose a different point below and create a kite.

Mathematics

- triangles are similar if they have the same angles
- in similar triangles corresponding sides are in the same ratio
- the ‘odd’ sides are parallel – so you have corresponding and alternate angles

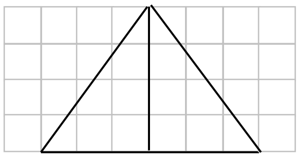
Other notes

Stress the parallelism of the ‘odd’ sides of isosceles triangles. The activity could be a preparation for work on the Theorem of Pythagoras. Use results to collect data for discussion.
This is an extension of C3.

C5 Area and the Rhombus

Task

On squared paper draw an isosceles triangle whose base is 6cm and whose height is 5cm.
Find its area.



Now draw another one below the base.
Describe the complete shape and find its area.
Do this for other base and height measurements.
Draw up a table for your results.

Adapting

Explore what happens when the base is twice the height or the height is twice the base.
Calculate the lengths of the sides of the triangle

Extending

What final shape do you get if the height of the second triangle is 1 cm longer than that in the first?
Derive a formula for the area if you know the lengths of the diagonals

Mathematics

- properties of isosceles triangle, rhombus, kite
- diagonals as perpendicular bisector or perpendiculars
- towards Pythagoras

Other notes

This is linked to C4, but the focus is more explicitly on area.

Area of triangle = ½bh

The two triangles have the same area so
the area of the rhombus = bh
b is the length of one of the diagonals of the rhombus and
h is half the length of the other diagonal.
The area of the rhombus is half the product of the diagonals.
Does this work for a kite?

C6 Plot a Set of Triangles

Task

On coordinate axes, where $0 \le x \le 8$ *and* $-6 \le y \le 8$. plot the points (2,1) and (8,1).
If this is taken to be the 'odd' side of an isosceles triangle, plot a third point to complete the isosceles triangle. Join up the points.
Create another isosceles triangle.
And another.
And another.

What is special about the points you are creating?
Why?
Where are the right angles? Why?

Adapting

Try the task with the points (2,1) and (6, 1) using different axes.
Try (2,1) and (2,8).

Extending

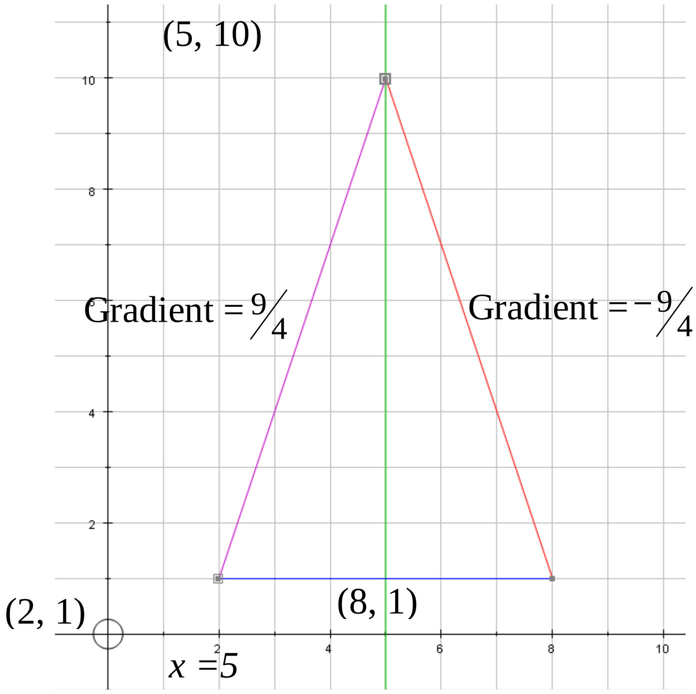
Try the task with the points (2,1) and (2, 7) using different axes.
Try (2,1) and (1,2)

Mathematics

- drawing axes
- plotting coordinates
- the line of symmetry will pass through the points plotted
- the line of symmetry is at right angles to the 'odd' side

Other notes

Stress that the points lie on the line of symmetry.
You may also wish to work on the equations of the line of symmetry and sides of the triangle.



What does this tell you about the angles the lines make with the x axis?

C7 Plotting and the Line of Symmetry

Task

Draw axes and draw some isosceles triangles so that $x = 6$ is the line of symmetry.

What is the connection between the coordinates at the ends of the 'odd' side?

If (2, 7) is one vertex of the triangle, what must one of the other points be? Try other points:
(3, 2), (-1, 4) (-11, 10) (93, 0) (a, b) (p, q)?

Adapting

Try this for isosceles triangles with lines of symmetry such as:

$$y = 2, x = -1, y = -3 \ldots$$

Extending

Try the activity with isosceles triangles with the line of symmetry

$$y = x, y = -x, y = x + 2 \ldots$$

Connect to the kite and the rhombus: ask pupils to construct these from given points and lines of symmetry

Mathematics

- drawing axes
- plotting coordinates
- line of symmetry as a line of reflection
- line of symmetry for the kite and the rhombus
- drawing the lines x=n, y=n
- generalising using algebra
- justification

Other notes

This task uses the same mathematics as C6, but the other way round. We believe that it is important to ask questions from different directions, as this calls for alternative approaches to the maths.

C8 Vectors and journeys

Task

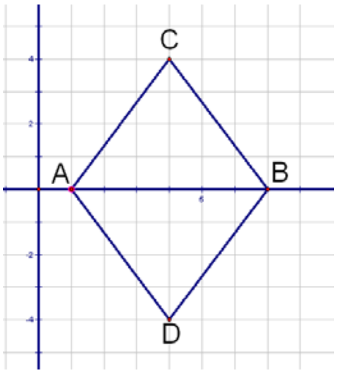
On a set of axes draw an isosceles triangle

A (1,0), B (7,0) and C (4, 4).

Which sides are equal?

Reflect the triangle in the odd side.

Describe point D which completes the rhombus.



Write down the vectors

AC, CA, AD, DA, BD DB, BC CB,

Write down all you know about the rhombus. Which vectors are the same and why? Relate what you know about the properties of the rhombus to make connections about the vectors

Adapting

Change the coordinates of the isosceles triangle.

When would your rhombus be a square?

What in the vectors helps you to justly this?

Extending

Use the vectors to find the lengths of the sides of the rhombus.

How would the vectors help you to find the gradients of the lines?

Find the equations of the lines.

6th form: give the rector equations of the lines

Mathematics

- properties of isosceles triangles
- column vectors
- equality of vectors
- magnitude of vectors (Pythagoras)

Other notes

Using the vectors to find the lengths of the sides of the rhombus is a different context for Y10 to practice Pythagoras.
The justification of the shape being a square is a context for working with scalar products in 6th Form.

As an alternative you could describe the drawing as:
- plot the point A (1, 0)
- translate the point A to B by $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$
- then translate B …

C9 Partitioning the Isosceles Triangle

Task

On squared paper draw axes for $0 \le x \le 10$ and $0 \le y \le 11$
Plot the points (2, 1) (10, 1) and (6, 11).
Join them up to form a triangle. How would you convince everyone that it is isosceles?
Plot and join the points (4, 6) (8, 6) (6, 1).

What shapes can you see?
If you measure one angle can you find all the others?
Which angles are the same? Why?
Find the area of each of the shapes.
How are the areas related?

Adapting

Create your own diagrams like this. (Why are mid-points so important?) Are the areas always related in the same way?

If you keep the base of the triangle 8cm and change the height, can you find a general formula for the areas in terms of the height?

Extending

Dividing in different ratios gives very different results.
On squared paper draw axes for $0 \le x \le 12$ and $0 \le y \le 13$
Plot the points (0, 1) (6, 13) and (12, 1).
Join them up to form a triangle.
Plot and join to form line segments:

1) (4, 9) (8, 9)
2) (8, 9) (4, 1)
3) (8, 1) (4, 9)

How many more triangles do you get? What different sizes?
How many angles do you need to measure to be able to find them all?
How are the parallelograms related to the isosceles triangles?

Mathematics

- properties of isosceles triangles, trapezium, kites, parallelograms.
- mid-points of line segments
- area of triangles
- parallelism
- similar triangles
- justification from geometrical properties

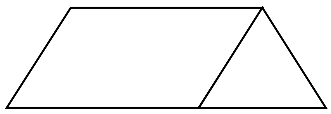
Other notes

Encourage the identification of the parallelograms in the task and identify parallel lines and related angles explicitly
This work can be related to enlargement and similar triangles.
The generalisation of the areas could be connected to trigonometry.
The coordinates were chosen to get the diagram in the first quadrant.

C10 Parallelograms with Triangles

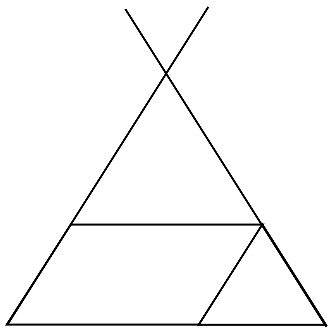
Task

On squared paper draw axes for $0 \leq x \leq 10$ and $0 \leq y \leq 10$
Plot the points (2, 1) (6, 1) (8, 4) and (4, 4).
Join them up to form a parallelogram. On the right hand side add an isosceles triangle.



Is there more than one that can be drawn?

Extend the outer sides upwards until they meet.



What sort of triangle is formed?
Why?
Which angles are the same?
Why?

Adapting

Draw your own parallelogram and add an isosceles triangle, then drawing the larger triangle.
Do you always get similar results?

Extending

Prove your results.

Find the areas of the triangles and parallelogram and the isosceles trapezium.

Mathematics

- properties of isosceles triangles, parallelograms.
- Angles between parallel lines
- area of triangles
- parallelism
- similar triangles
- justification and proof

Other notes

Column vectors might be useful to prove lengths are the same

The sides of the parallelogram can be represented by either of these column vectors. These column vectors represent lengths of the same size. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$

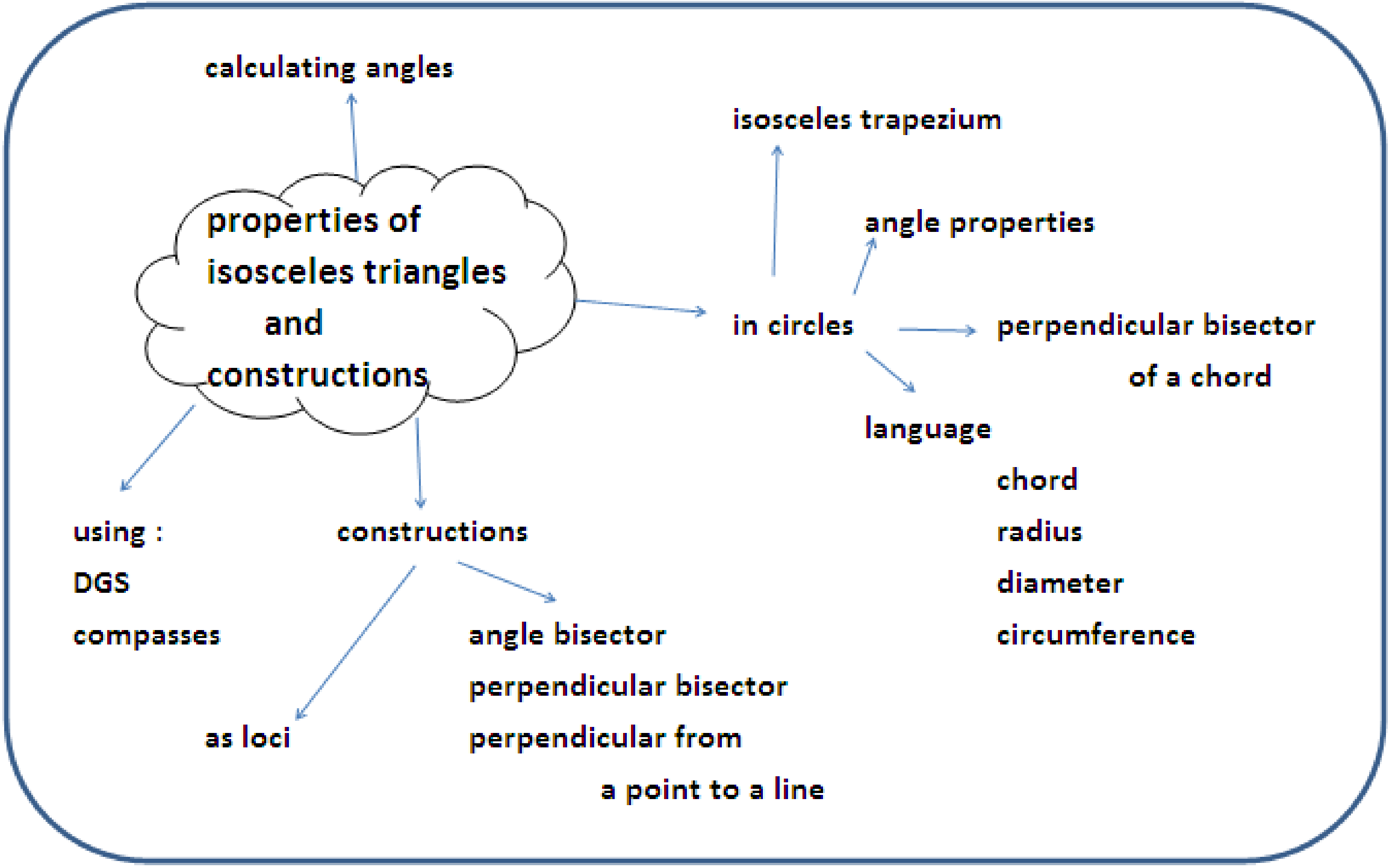
Other vectors which are the same length are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Section D: Constructions

Section notes

In this section we offer some construction tasks using the properties of the isosceles triangle and in particular the powerful line of symmetry, which is both perpendicular bisector and angle bisector. There are some well known constructions with slight variations on the 'normal' instructions – the angle bisector, a perpendicular from a point to a line, a perpendicular bisector and so on. If you know about the properties of the isosceles triangle most of the constructions relate to these properties and so can be deduced. We think connecting in this way demands less rote learning from the students about 'what-to-do'.

From various sessions we have run with teachers we have played with some alternatives to the well known construction likely to be found in a text book. The isosceles trapezium in concentric circles was a revelation. We hope you enjoy this one.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D1 Circle a Rectangle

Task

On squared paper draw a large rectangle.
Draw the diagonals.
Use the point where the diagonals meet as the centre to draw a circle which passes through the vertices of the rectangle.

Find any isosceles triangles
Find all the angles – how many do you need to measure?

Adapting

Try this with other rectangles.
What is the minimum number of angles you need to measure in order to calculate the other angles.
How might you convince others of this?

Extending

Is it possible to construct a circumcircle around any rectangle?
Why?
Is the intersection of the diagonals always the centre of this circumcircle?
Why?
Find the area of the isosceles and right angled triangles.

Mathematics

- isosceles triangles
- rectangle properties
- constructing a circle given centre and radius as a line segment
- diagonals of a rectangle bisect each other
- radii of a circle are of equal length
- the four vertices of the rectangle are equidistant from the point where the diagonals intersect

Other notes

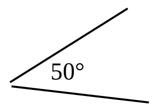
The diagonals of the rectangle bisect each other giving four isosceles triangles.
When a circle can be drawn around a shape so that all vertices of the shape lie on the circumference it is called a circumcircle.
Opposite angles of a cyclic quadrilateral sum to 180º. The rectangle is a special case of the cyclic quadrilateral.
The angle in a semi-circle is a right angle; this construction allows you to do initial work.

D2 The Isosceles Triangle from the odd angle

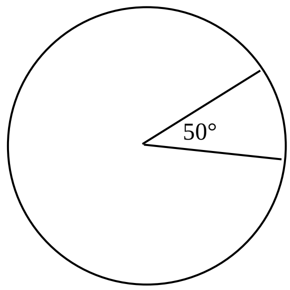
Task:

To draw a triangle with equal sides of 6cm and the non-equal angle of 50°.

Draw an angle of 50°. Make sure the arms of the angle are longer than 6cm.



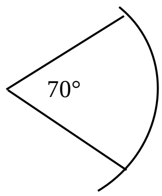
Construct a circle of radius 6cm, centre the vertex of the angle.



Complete the triangle by drawing the chord.
How do you know that the triangle is isosceles?
How would you convince others of this?

Adapting

Draw a triangle with equal sides of 6cm and the non-equal angle of 70°, but instead of drawing the whole circle only draw the minor arc.



Try this for an angle of 120º, 90º, 53º…

Extending

Use this method of radii and arcs to construct an isosceles triangle where the equal sides are 6cm.
Find the length of the third side.
Change the construction so that the third side is 8cm.

Mathematics

- language of radius, chord, sector
- radii, equality and the sides of the isosceles triangle
- recognising the sides of the isosceles triangle as radii

Other notes

The transition from circle to arc needs to be worked on carefully. Some pupils find it difficult to see the equality of radii in circles. Many more do not see radii in arcs so the equality is harder for them to see.
Make sure that reasoning is articulated by all pupils (with or spoken.)
Changing the construction from being given the angle to being given the third side is a leap (and not how you might usually start) but it extends from the previous constructions by use of a ruler or circle or arc.

D3 Isosceles Triangles and the Rhombus

Task:

Use a large sheet of paper, begin drawing in the middle.
Draw a line segment 6 cm long, label the ends A and B.
Draw a circle, centre A, radius 8cm.
Draw a circle centre B, radius 8cm.
Mark the points where the circles cross – label the point above the line C and the point below the line C'.
Join the points ACBC'A.

- ABC and ABC' form isosceles triangles. Why?
- What shape is ACBC'?
- Convince someone that the shape has to be a rhombus.
- Which line segments show the lines of symmetry of the rhombus? Why is one line of symmetry of the rhombus also a line of symmetry for the two isosceles triangles?

- What property of the lines of symmetry means that CC' must be the bisector of AB and that it is perpendicular to AB?

Adapting

Use different radii to construct the circles, for example 7cm, 10cm, 5cm ...
What radius gives an equilateral triangle?
What radius gives a triangle of half the height of the equilateral triangle?

Extending

Change the construction so that you use arcs rather than complete circles.
Find the area of each rhombus you have constructed.

What happens if the two circles have different radii?

Mathematics

- use of compasses
- properties of shapes, including symmetry
- the perpendicular bisector.

Other notes

Points to ponder:

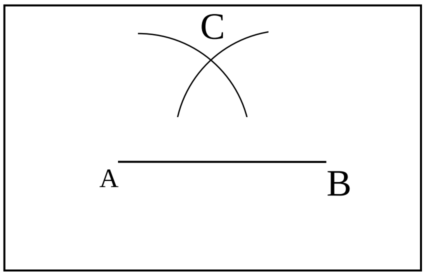
- The locus of a point a given distance from a fixed point is a circle.
- What does the intersection of the circle/arcs have to do with points being equidistant from other points?
- If a shape has exactly two lines of symmetry, these lines must bisect each other.
- If a shape has exactly two lines of symmetry, these lines must be perpendicular to each other.

For area, you could do the constructions on centimetre squared paper.

D4 Nesting Isosceles Triangles

Task:

Use a large sheet of paper, begin drawing in the middle.
Draw a line segment 10cm long.
Label the ends of the segment A and B.
Draw a small arc of a circle above AB, centre A, radius 13cm.
Draw a small arc of a circle above AB, centre B, radius 13cm.
Mark the point where the arcs cross – label the point C.
Join the points ACB.



Construct point D on the same diagram using arcs of radius 10cm.
Join ADB.
Construct point E on the same diagram using arcs of radius 8 cm.
Join AEB.
Construct point F, G and H using your own choice of radii.
What do you notice about points C, D, E, F, G and H?
Could one of the points that belong to this set lie on the line segment AB? If so, where?

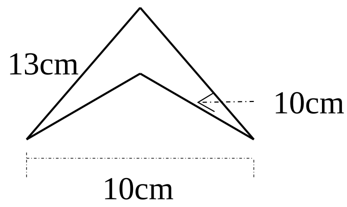
Adapting

Construct some isosceles triangles on the other side of the line segment.
Find rhombuses and kites in the diagram.
Describe the construction for a rhombus and a kite. What is different? Why?

Extending

Find symmetrical arrowhead quadrilaterals in your figure.
Explain how they have been constructed.

Construct an arrowhead with the sides 13cm, 13cm, 10cm and 10cm and a maximum width of 10cm.



Is the arrowhead defined in this way unique? Why?

How could the construction of the isosceles triangle help you to construct a right-angled triangle of base 6cm and hypotenuse 12cm?

Mathematics

- use of compasses
- properties of the isosceles triangle, including symmetry
- properties of compound shapes
- the perpendicular bisector
- the line of symmetry as angle and perpendicular bisector.

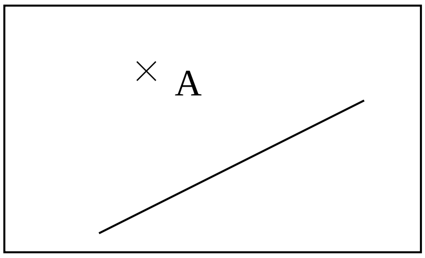
Other notes

Why do the points C, D, E, F etc. lie on a straight line? – how might you exploit this to work on pupils drawing accuracy?
Why does this line meet the line segment AB at right angles?
Compare this method of constructing a perpendicular bisector with the more usual method of arcs above and below the line segment. Are there any occasions where this might be useful?
Why do you need TWO isosceles triangles?
How different would the work look if you used circles rather then arcs?

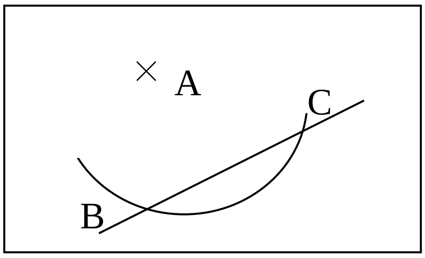
D5 An Isosceles Triangle – from a point to a line.

Task:

Construct an isosceles triangle with one side on part of the given line segment and a vertex at the point A.



Draw an arc of a circle, with A as the centre, so that the arc cuts the line segment in two places, B and C.



Join AB and AC

Why is triangle ABC an isosceles triangle?
Try some other points and line segments: does this method always give an isosceles triangle.

Adapting

With B and C as centres draw two arcs, using the same radius, so that they intersect.
Label the point of intersection D.
What shape is ABDC?
How would you convince someone of this?

Construct D so that the shape is a rhombus.

Extending

Try constructing several different isosceles triangles. Use the information to help you to try to construct B and C so that the triangle is equilateral.

Construct B, C and D so that the shape is a rectangle.

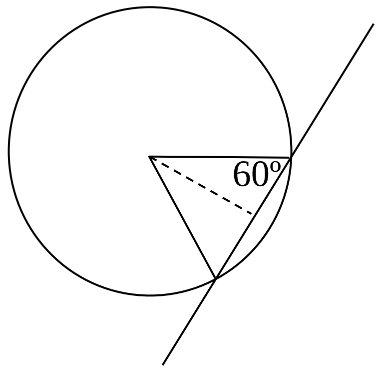
Mathematics

- use of compasses
- properties of the isosceles triangle, including symmetry
- the locus of points which make a circle.

Other notes

Connect this construction to the isosceles triangle in the circle (Task 2).
Make the connection to radii explicit.

Finding the equilateral triangle is not straightforward, but the calculation could make a challenge for your bright Y10s.

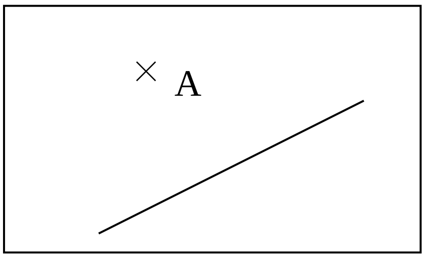


D6 An Isosceles Triangle – dropping Perpendicularly from a Point to a Line

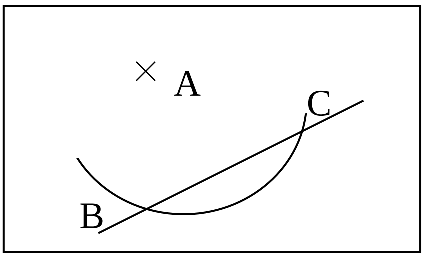
Task

To drop a perpendicular from a point A to the line segment,

- draw a line segment and a point



- draw an arc of a circle centre A that cuts the line segment in two places



Why do the points ABC form an isosceles triangle?

Construct another isosceles triangle, with BC as the non-equal side, on the other side of the line segment.

Why is AD perpendicular to BC?
What does this have to do with the symmetry of the isosceles triangle?

Adapting

Use the construction to form a rhombus with A as a vertex and the line segment as a line of symmetry.

Draw a set of rhombuses with A as a vertex and the line segment as a line of symmetry.

Extending

Draw a triangle and drop a perpendicular from one vertex to the opposite side. Use this to find the area of your triangle.
Try a diagram when the vertex is at the corner of an obtuse angle.
Draw a triangle and drop perpendicular from each vertex. What is special about the point where the three lines cross?

Mathematics

- use of compasses
- properties of the isosceles triangle, including symmetry
- the line of symmetry being perpendicular to the odd side
- the locus of points which make a circle.

Other notes

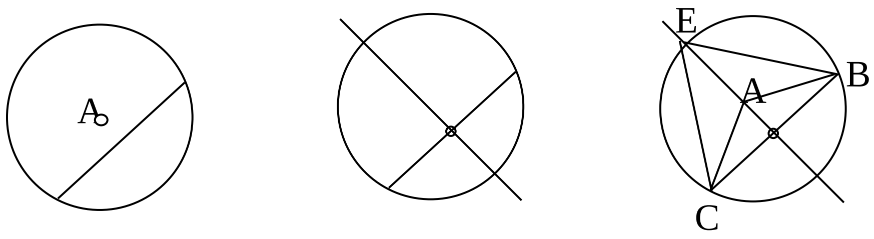
The image of the isosceles triangle in the circle needs to be used (Task D2).
Link explicitly to Task D5.
Use the construction as a route to the circle theorem

- the perpendicular bisector of a chord passes through the centre of the circle.

D7 Triangle on the Chord (using DGS).

Task

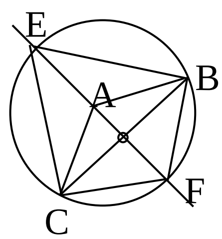
Using a DGS draw a circle centre A, point on circumference B.
Draw a line segment BC with its ends on the circle (a chord).
Mark the mid-point.
Draw a perpendicular to the chord through the midpoint.



Join the ends of the chord to the point on the major arc where the perpendicular intersects.
Why is the triangle BCE isosceles?
Join the ends of the chord to the centre of the circle.
Mark and measure the angles BCE, CBE, CEB and CAB.
As you drag- what do you notice about the connection between angle CEB and CAB?

Adapting

Join up the chords to the fourth point to make a kite.



As you drag compare angles CEB,CFB and CAB.

What is the size of angles ECF and EBF? Why?

Extending

On a sketch, if angle CEA is 40° calculate all the other angles in the figure.

For a class that is comfortable with the isosceles triangle and DGS get them to sort out the task of constructing the isosceles triangle using DGS and the first givens of a chord and a circle.

Mathematics

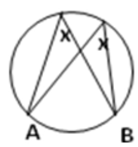
- properties of the isosceles triangle: perpendicular bisector as the line of symmetry
- working towards the connection between the angle at the centre and the angle at the circumference.

Other notes

There are several theorems about angles in circles that emerge here and in other tasks. All of these can be deduced from the properties of isosceles triangles The main three are:

- the angle in the semicircle, or the angle at the circumference subtending the diameter, is a right angle

- If angles at the circumference stand on the same arc of the circle they are equal

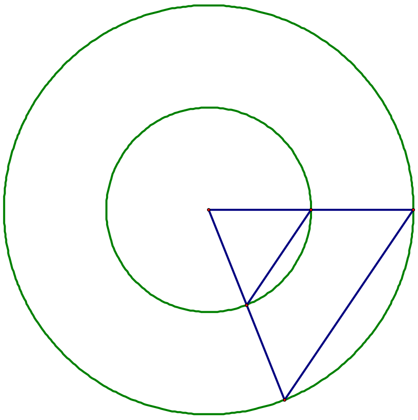


- The angle at the centre of the circle is double the angle at the circumference if they stand on the same arc

D8 Isosceles Triangle to Isosceles Trapezium.

Task

Draw two concentric circles of radii 4 and 8 centimetres.
Draw two radii and the two chords
Find the isosceles triangles



Find the angles.
How many angles do you need to measure?
How do you know that the other shape is an isosceles trapezium?

Adapting

Measure the lengths of the sides.
Try other trapezia.
How do the radii of the circles affect the ratio of the sides of the isosceles triangles and the lengths of the sides of the trapezium?
Use the construction to draw an isosceles trapezium with equal sides of length 3cm.

Extending

Use the construction the draw an isosceles trapezium with equal sides of length 6cm and parallel sides of length 8cm and 12cm

Mathematics

- use of compasses
- properties of the isosceles triangle and isosceles trapezium
- the line of symmetry being perpendicular to the odd side
- the locus of points which make a circle.
- the perpendicular bisector of a chord passes through the centre of the circle.

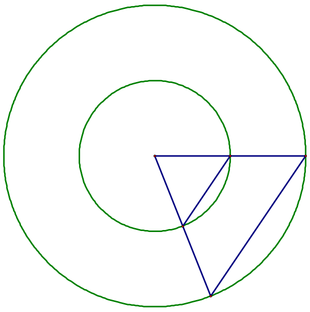
Other notes

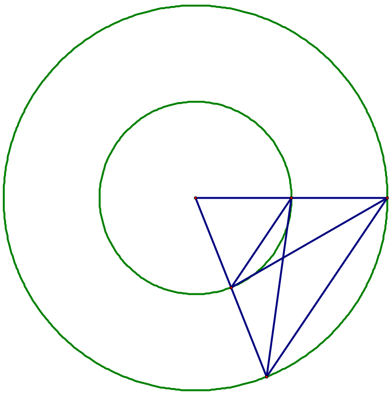
The image of the isosceles triangle in the circle needs to be used (Task D2).
Link explicitly to Task D5.
Use the construction as a route to the circle theorem
An isosceles trapezium is a trapezium which has the two non-parallel sides equal.

D9 The Isosceles Trapezium and its Diagonals

Task

Draw two concentric circles of radii 4 and 8 centimetres.
Draw two radii and the two chords
Where are the isosceles triangles and an isosceles trapezium?



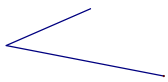


Draw the diagonals of the trapezium
Find the angles.
How many angles do you need to measure?
Which angles are the same?

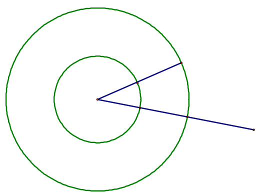
Where is the kite?
Draw its line of symmetry.
Which angle(s) does it bisect?

Adapting

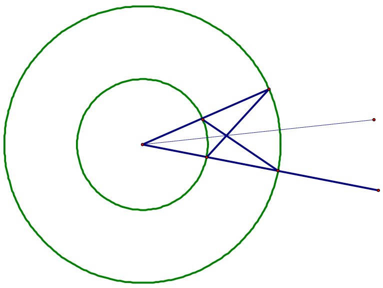
Use the method to bisect an angle.



- using the vertex of the angle draw two concentric circles (or arcs)
- draw lines between the intersections of the circles and line segments

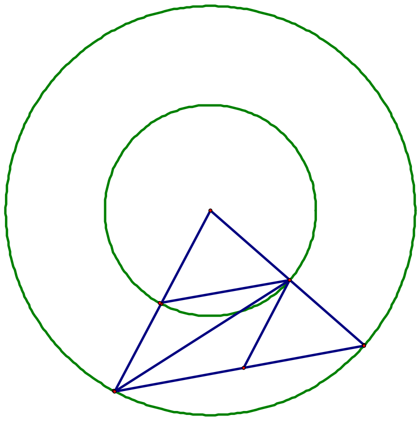


- draw a line from the vertex to the intersection of the diagonals.



Extending

Using only one diagonal of the trapezium, construct the diagram.
From one end of the diagonal an isosceles triangle is constructed.
What angles can be found?
What shapes can be found?



Mathematics

- use of compasses
- properties of the isosceles triangle, isosceles trapezium kite.
- the line of symmetry of a late bisects two angles
- angles between parallel lines.

Other notes

This is a very different construction from the normal angle bisector construction.
Exploit the knowledge of isosceles triangles to explore

- parallelism
- alternate, corresponding, supplementary angles
- properties of similar triangles

D10 Isosceles Triangle Patterns.

Task:

Draw a circle – use a large radius.
Measure angles of 45° around the centre and draw in the radii (use a light pencil line).

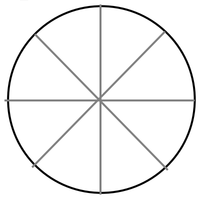


Figure 1

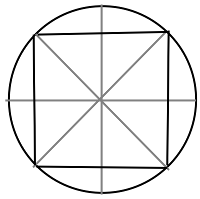


Figure 2

Join alternate points on the circumference.
What shapes have been constructed?
How many right-angle isosceles triangles are there in the figure?
Join up the other points on the circumference and shade the new triangles (figure 3).

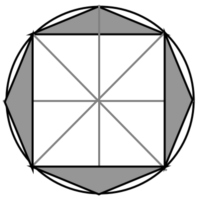


Figure 3

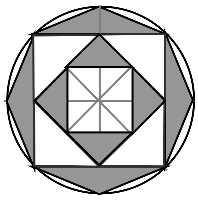


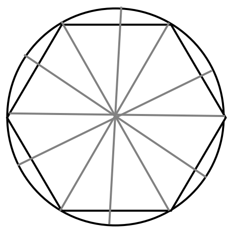
Figure 4

Calculate the angles of the new triangles.
Continue the construction by joining up alternate points to create new triangles, shade in alternate colours.
Find the area of each of the shapes.
Describe the relationships between the areas.

- identifying embedded shapes
- area and right-angles triangles

Adapting

Repeat the construction beginning with angles of 30°.



Extending

Describe what will happen with different starting angles – check by constructing.

Mathematics

- use of angles
- identifying shapes by certain properties
- similar triangles

Other notes

Link the image to the construction of regular polygons.
How do you help with inaccuracy?

Section E: Reflections and Rotations

This section is split into two sub-sections with the last activity combining reflection and rotation.

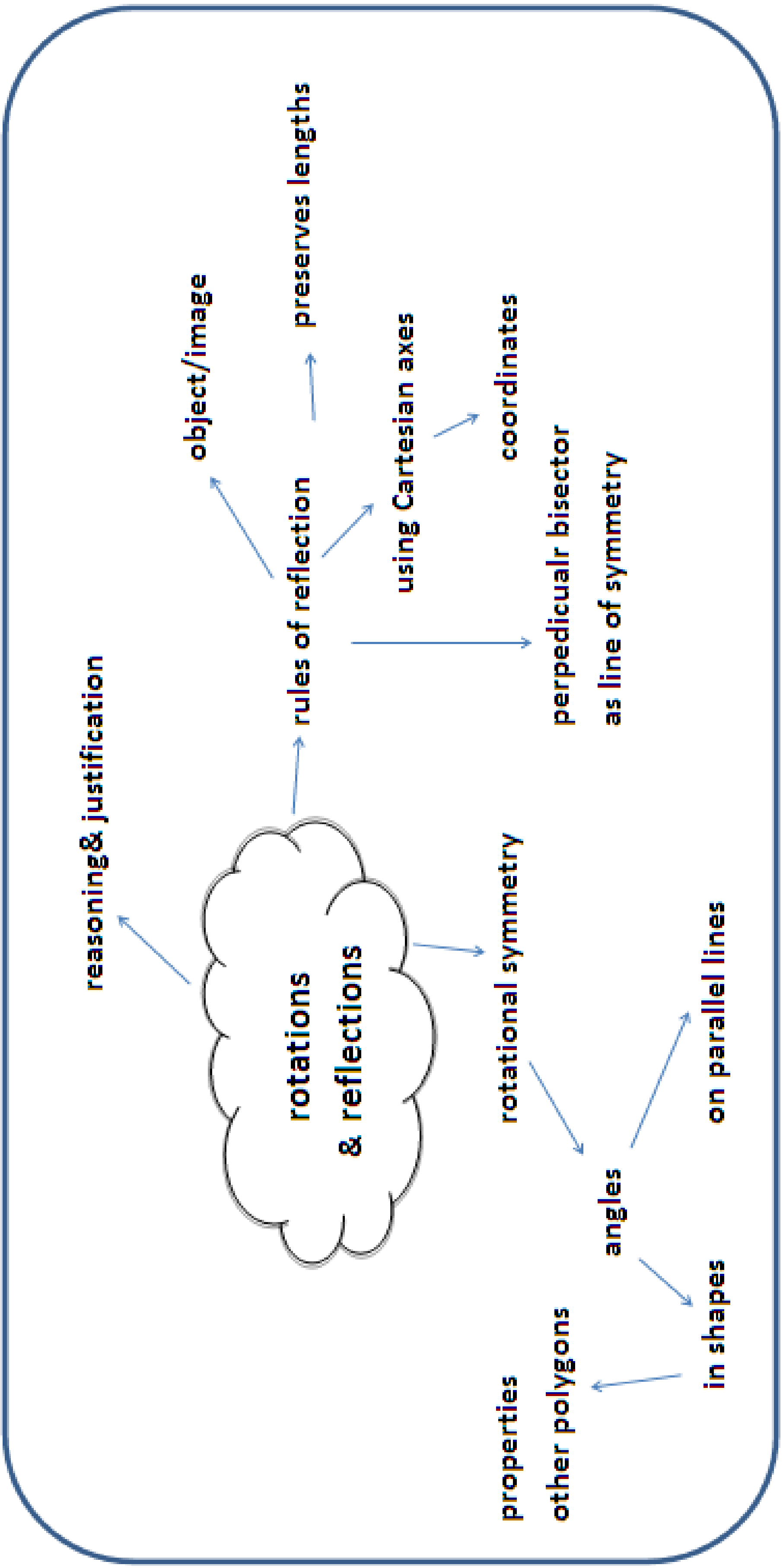
Section notes

In this section we offer some activities that connect the isosceles triangle to some aspects of transformation geometry, namely reflections and rotations. There is a range of mathematics that can be practised and hence connected. Some of the tasks have as an extension the possibility of justification and proof.

Many of these tasks use coordinates and a Cartesian grid and so give practice for drawing axes, plotting points and drawing graphs of linear equations. Rotating or reflecting isosceles triangles gives new polygons to be investigated – lots of measurement and/or calculation of lengths and angles.   There are things to be noticed about the consequences of transforming a shape by rotation or reflection – what is the difference between the objet and the images, which lengths and angles are preserved, and so connections to congruence and similarity.

Two of the tasks use a dynamic geometry software (DGS) – the screen dumps shown are from Geometers Sketchpad. The animation available allows points to vary easily (by dragging) and so lots of special cases can be noticed that might help with generalisation.

Task E8 uses a geostrip – you will probably find some in a cupboard!


rotations
& reflections
reasoning& justification
rules of reflection
object/image
preserves lengths
using Cartesian axes
coordinates
perpedicualr bisector
as line of symmetry
rotational symmetry
angles
on parallel lines
in shapes
properties
other polygons

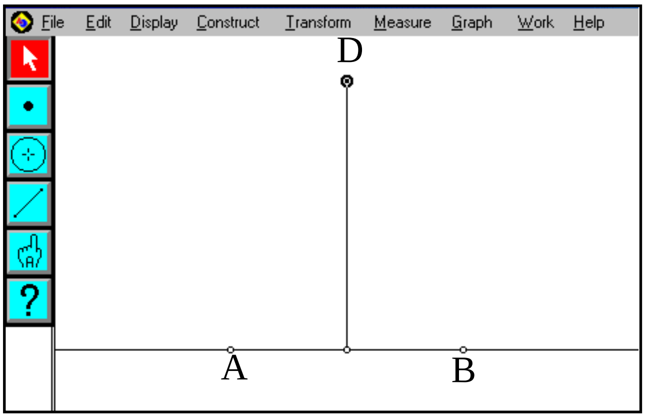
Reflections

E1 Reflecting Isosceles Triangles: Angles and Polygons (DGS)

Task

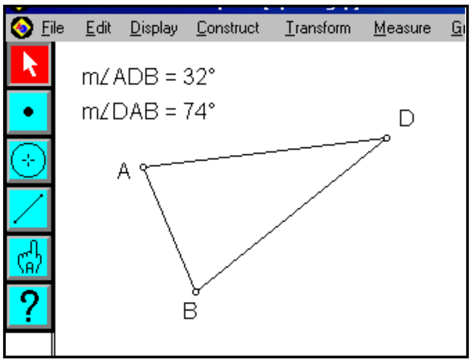
Construct an isosceles triangle by drawing:

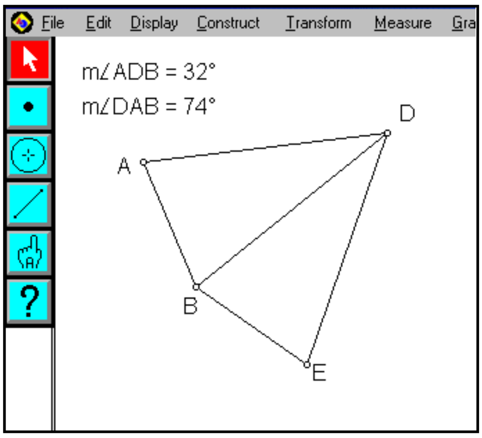
- a line segment to act as the line of symmetry,
- a line perpendicular to the line segment
- a point on this line
- reflect the point in the first line segment
- use the three points as vertices of the triangle



Measure the angles and hide the construction lines.

Reflect point A in the line DB, to get the vertex of a new isosceles triangle.

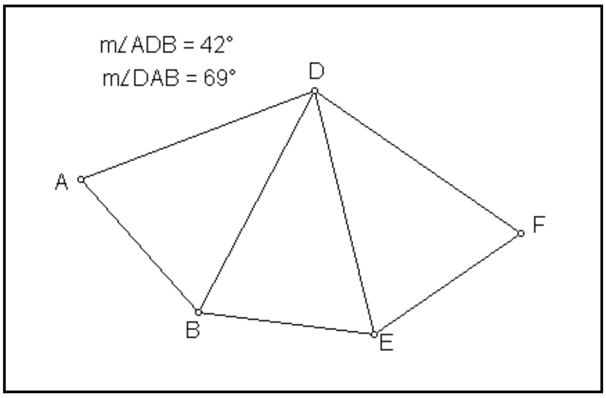




Why is it isosceles?
Drag point A. What happens?

Adapting

Reflect point B in the line DE, to obtain another isosceles triangle.
Drag point A.
What shapes can you obtain?
What is angle ADB if ADF is a straight line?



Extending

Keep reflecting the triangle until there is an overlap of the last triangle over the first.

- change the position of D until you get a polygon.

What polygons can you make?
Find the connection between the number of isosceles triangles and the angle ADB.

Mathematics

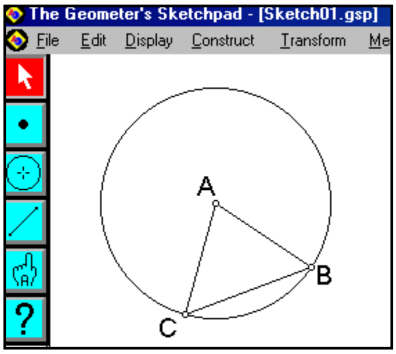
- reflection
- under reflection, the object and image are equidistant from the mirror line
- recognising the isosceles triangles and its equal angles
- the perpendicular bisector as the line of symmetry of the isosceles triangle
- properties of quadrilaterals and other polygons
- using the properties of reflection in geometrical reasoning

Other notes

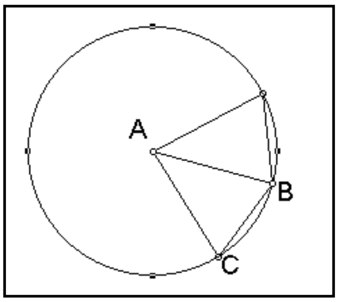
A kite is produced from the two triangles, with the mirror line being a line of symmetry. The diagonals of a kite cross at right angles, as do the diagonals of a rhombus and a square. (A square is a special kind of rhombus and a rhombus is a special kind of kite.)
If ABF is a straight line the angles of each triangle are 60º - the equilateral triangle being a special case of the isosceles triangle.
Increasing the number of triangles could lead to work on regular polygons.

E2 Isosceles Triangles in a Circle (DGS)

Task



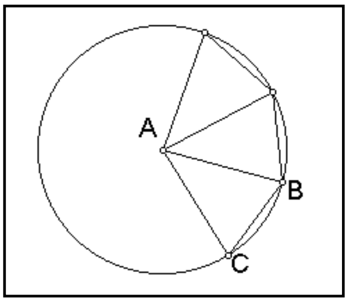
Construct an isosceles triangle by using a circle (centre and point on circumference).
Construct a point on the circle.
Join the three points by line segments.
Why is the shape an isosceles triangle?
How can you convince others?



Reflect the point C in the line AB, to obtain a vertex for a new isosceles triangle.
Why is the new shape an isosceles triangle?
Does the image of C lie on the circle? Why?

Drag the point C.
What happens?
Describe the shapes formed.

Adapting



Reflect the image of C to form a third triangle.
Drag C.
Describe the shapes you have made.
What happens if AC meets the new triangle?
Find the angles.

Do you ever get any pairs of parallel lines?
How can you convince others that they are parallel.

Extending

Animate the construction so that C moves around the circle. Ask pupils to watch. Stop the animation and ask pupils to:

- write a description of what they have seen
- describe and name the various shapes
- draw sketches of what they have seen
- discuss with a partner and come to a shared description of what they have seen

Mathematics

- angles in an isosceles triangle
- reflection preserves lengths and angles
- the angle at the centre of the circle is twice the angle at the circumference (special case)

Other notes

Unlike the previous task, the symmetry of the isosceles triangle is not made explicit in this task. The symmetry is implicit in the diameters being lines of symmetry of the circle.

E3 Reflect a Point (squared paper)

Task:

Draw a set of axes $0 \leq x \leq 10$ and $0 \leq y \leq 10$
Draw the line $x = 6$ and plot and label the point (6, 9) A.
Plot the point ((2, 1) and label it B.
Reflect B in the line $x = 6$ and label the image point C.

Join the point A, B and C.
Describe the triangle ABC.
Describe how would you prove to someone what type of triangle ABC is?

Plot the point (3, 3) and label it D.
Reflect D in the line $x = 6$ and label the image point E.

What connection is there between the line segments BC and DE? Why?

Find all pairs of angles in the diagram that are equal.

Adapting

On new axes, draw the line $x = 6$ and plot and label a point A on the line.
Plot and label a point B not on the line.
Reflect B in the line $x = 6$ and label the image point C.
Join the point A, B and C.
If you get a triangle, describe the triangle ABC. (Why might you not get a triangle?)
Describe how would you prove to someone what type of triangle ABC is?

Choose any point on the line AB and label it D
Reflect D in the line $x = 6$ and label the image point E.

What connection is there between the line segments BC and DE? Why?

Try the same task with the line y = 5

Extending

Choose a point B so that triangle ABC will be right-angled.
Choose your own line and plot a point on the line and label it A.
Plot and label a point B not on the line.
Reflect B to obtain an image point C etc.

Mathematics

- reflection
- recognising the isosceles triangles
- practising plotting points and using Cartesian axes
- using the properties of reflection in geometrical reasoning
- recognising parallel lines
- finding equal angles in an isosceles triangle, corresponding angles in parallel lines

Other notes

Reflection preserves shape and size.

The mirror line perpendicularly bisects BC – it is the line of symmetry of BC.
The mirror line passes through A, so the mirror line must be the line of symmetry of triangle ABC.
If a triangle has one line of symmetry it is isosceles.
If BA is perpendicular to the mirror line you will not get a triangle.

E4 Reflect a Point in y = x (squared paper)

Task:

Draw a set of axes $0 \leq x \leq 10$ and $0 \leq y \leq 10$
Draw the line y = x and plot and label the point (8, 8) A.
Plot the point ((4, 0) and label it B.
Reflect B in the line y = x and label the image point C.

Join the point A, B and C.
Describe the triangle ABC.
Describe how would you prove to someone what type of triangle ABC is?

Plot the point (6, 4) and label it D.
Reflect D in the line y = x and label the image point E.

What connection is there between the line segments BC and DE?
Why?

Find all pairs of angles in the diagram that are equal.

Adapting

Draw a set of axes $0 \leq x \leq 10$ and $0 \leq y \leq 10$
Draw the line y = x and plot and label the point (0, 0) A.
Plot the point ((4, 8) and label it B.
Reflect B in the line y = x and label the image point C.

Join the point A, B and C.
Describe the triangle ABC.
Describe how would you prove to someone what type of triangle ABC is?

Plot the point (1, 2) and label it D.
Reflect D in the line y = x and label the image point E.

What connection is there between the line segments BC and DE?
Why?

Find all pairs of angles in the diagram that are equal.

What is the equation of the line that passes through A and B?
What is the equation of the line that passes through A and C?

Describe any connection between the equations of the two lines.

Extending

Draw axes and the line y = *x*.
Let A be (0, 0) and B be (8, 2) – reflect B in the line to obtain C.
What type of triangle is ABC?
Find the equations of the line AB and AC – what connections are there?

Try the points:

A (0, 0) B (10, 2)
A (0, 0) B (3, 9)
A (0, 0) B (-2, 6)
A (1, 1) B (5, 9)
A (1, 1) B (9, 3)

Mathematics

- reflection
- practising plotting points and using Cartesian axes
- recognising the isosceles triangles
- recognising perpendicularity
- the line of symmetry of the isosceles triangle perpendicularly bisects one side and bisects one angle.
- using the properties of reflection in geometrical reasoning

Other notes

The nature of this task means that coordinates have to be chosen with care.

E5 Reflect Two Points (squared paper)

Task:

Draw a set of axes – plan ahead before you decide the scale.
Draw the line y = 4.
Plot two points A and B so that the line segment that joins them is NOT parallel to y = 4.
Reflect the two points in y = 4.
Join the original two points and extend the line segment so that it meets y = 4, label this point C.
Do the same for the image points.

What happens?
Why?
Describe the triangles ACA' and BCB'.

Are they similar? Why?
Which angles are the save?

Adapting

Try this with other points and different lines.

Extending

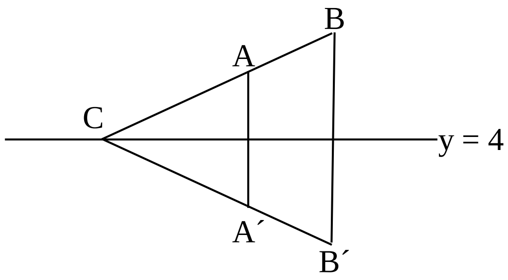
Use 3 or 4 points that lie on a line not parallel to the mirror line.
Find the equations of the lines AA´, BB´, AB, A´B´

Mathematics

- reflection
- practising plotting points and using Cartesian axes
- recognising the isosceles triangles
- recognising perpendicularity and parallelism
- angles on parallel lines
- the line of symmetry of the isosceles triangle perpendicularly bisects one side and bisects one angle.
- using the properties of reflection in geometrical reasoning

Other notes

This activity could begin to allow work on similar triangles.



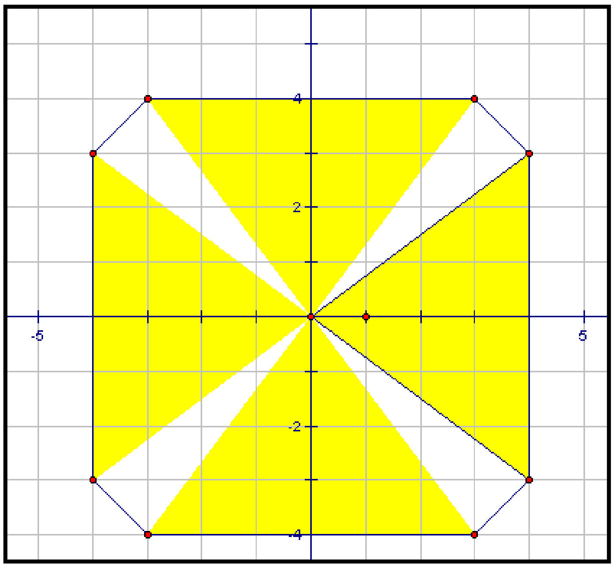
You can work explicitly on alternate, corresponding and supplementary angles.

E6 Triangle Reflection - towards pattern (squared paper)

Task:

Draw a set of axes $0 \leq x \leq 5$ and $0 \leq y \leq 5$
Plot the points (0, 0) (4, 3) and (4, -3). Join the points. Describe the triangle (angles, lengths, symmetry.)

Reflect the triangle in x = 0 (the y axis).
Reflect the triangle in $y = x$.
Reflect the triangle in $y = -x$



Join the vertices of the triangles to form an octagon.
Describe the other triangles in the shape.
Describe the symmetry of the octagon.
Explain the symmetry.

Adapting

Do this with other triangles, by changing (4, 3) and (4, -3).

Extending

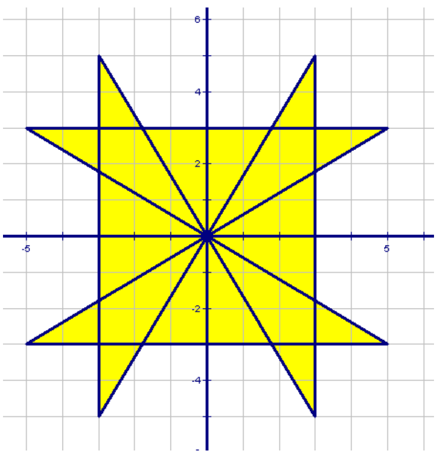
Try doing this activity using a graphical calculator or a graphical package such as Autograph or Omnigraph.
Change the mirror lines to ***y* = -2*x*** and ***y* = ½*x***
Experiment with other mirror lines.

Mathematics

- reflection
- coordinates and straight line graphs
- recognising the isosceles triangles
- congruence
- parallel lines
- symmetry and the octagon
- using the properties of reflection in geometrical reasoning

Other notes

What triangles give you an octagon?
You may prefer to give the coordinates when adapting.
You may wish to work with overlapping shapes, finding hidden triangles.



Rotation

E7 Rotate a Line Segment (squared paper)

Task:

Draw a set of axes $0 \le x \le 8$ and $0 \le y \le 10$
Plot and label the points (1, 5) A and the point (6, 5) B.
Join the points.
Rotate the line segment 90° anticlockwise about A.
Label the image of B as C.

Join the point A, B and C.
Describe the triangle ABC.
Describe how would you prove to someone what type of triangle ABC is.

Now rotate the line segment 90° anticlockwise about B.
Label the image of A as D.
Describe the triangle BAD.
Describe how would you prove to someone what type of triangle BAD is?

Find all the angles in the diagram that are equal.

Triangle ABC has one line of symmetry.
Triangle BAD has one line of symmetry.
Why does ACBD have no lines of symmetry?

Adapting

Try different angles of rotation.

Extending

Find the angle(s) of rotation so that ACBD is a rectangle.

Find the angle(s) of rotation so that ACBD has a line of symmetry.

Mathematics

- rotation
- recognising the isosceles triang les
- coordinates, Cartesian axes, straight line graphs
- using the properties of rotation in geometrical reasoning
- recognising parallel lines
- rotational symmetry of a parallelogram.
- finding equal angles in an isosceles triangle, corresponding angles in parallel lines.

Other notes

Rotation preserves shape, size and sense.

Most students assume that the parallelogram has a line of symmetry – use the isosceles triangles to work on why the quadrilateral does not have a line of symmetry.
Work on the rotational symmetry of the parallelogram – the centre of rotational symmetry is the mid-point of AB.
A DGS might be useful.
Autograph draws circles when doing rotations which may be useful.

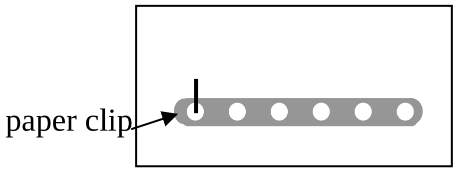
E8 Rotate a Geostrip

Task

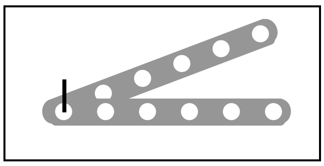
You need a long geostrip (or a piece of card with holes punched in) a large piece of paper and a paper clip (some blu-tak could be useful).

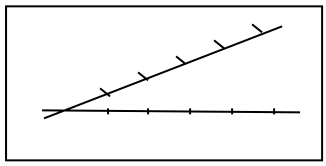
Open up the paper clip to make one piece at right angles to the plane of the rest

Place your geostrip on the paper and punch the paper clip through the paper from underneath through an end hole.

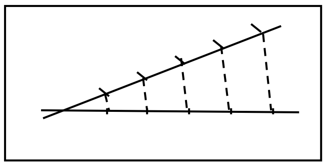


Mark the centre of each other hole.
Rotate the geostrip around the paper clip and mark the new positions of the holes.



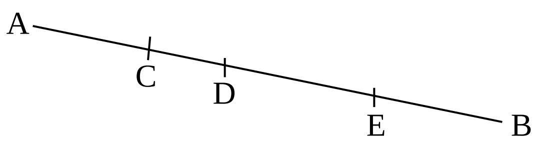


Draw the lines that join the holes to the position of the paper clip.
Join each point to its image point.
Measure the angles.
Where are the isosceles triangles?

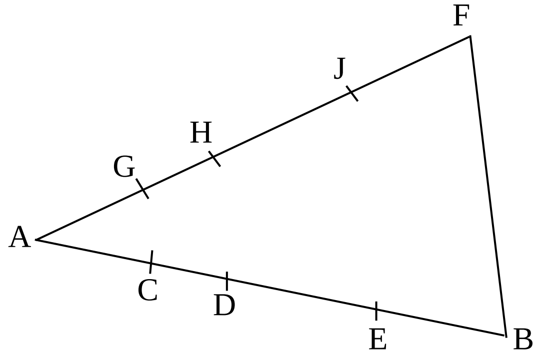


Adapting

Try this again but rotate about a different hole.
Or repeat by drawing a line segment AB and mark some points C, D, and E anywhere between A and B.



Make a tracing so that you can rotate the line segment 50° anticlockwise about A.
Label the image of B as



F.
Join the points A, B and F.

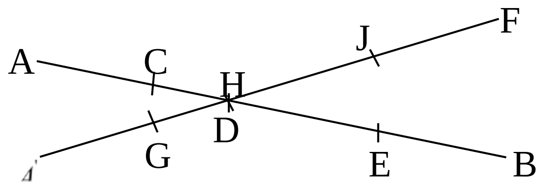
Describe the triangle ABC.

Describe how you would prove to someone what type of triangle ABF is.

Label the image of C as G, the image of D as H and the image of E as J.
Join C to G, D to H and E to J.
What is the relationship between the line segments CG, DH, EJ and BF? Why?

Extending

Try different angles of rotation.
Prove that the lines joining object and image are parallel.
Try different centres of rotation, B, C, D or E



Mathematics

- rotation
- recognising the isosceles triangles
- using the properties of rotation in geometrical reasoning
- recognising parallel lines
- relationship between rotation, isosceles triangles and parallel lines
- parallelism and similar triangles
- finding equal angles in an isosceles triangle, corresponding angles in parallel lines.

Other notes

Rotation preserves shape, size and sense.
Geostrips allow an easier but less accurate way of working on the task.
You could work with a DGS or a graphical package.

E9 Rotate a Triangle

Task

Construct a right-angled isosceles triangle.
Using the vertex at the right angle as your centre, rotate the triangle 45º.
Rotate that triangle 45º.
Repeat until you have all the unique positions.
How many rotations would take you back to the beginning?
Why?
Describe the shape, including its symmetry
How many isosceles triangles are there in the shape?
What other shapes are there?
Can you draw a circle that passes through the outside points?
Explain why.

Are any other circles related to the shape?

Adapting

Try rotating 60º.
Try other angles.

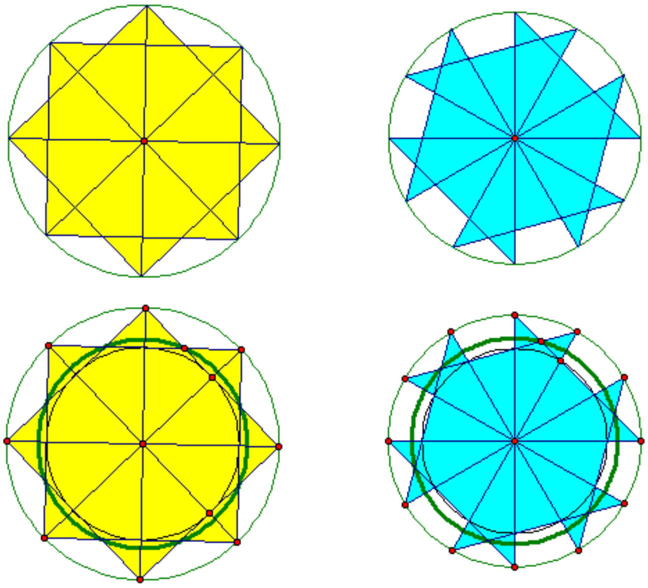
Extending

Relate the area of the final shape to the area of the original triangle.
Try other types of isosceles triangles.

Mathematics

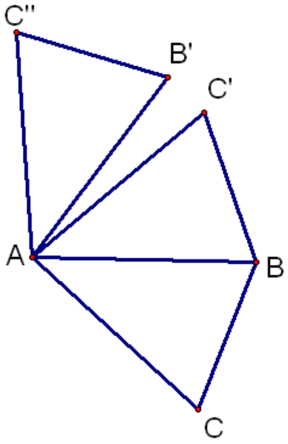
- construction – perpendicular, equal sides
- symmetry, line and rotational
- rotation
- properties of isosceles triangles

Other notes



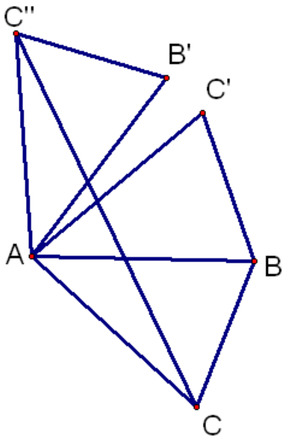
The inner circle on the 45º rotation offers illustration of an incircle and its related tangents.

E10 Reflecting and Rotating



Task

Draw an isosceles triangle ABC, where AB = AC and angle BAC is 30º .
Reflect the triangle in AB to get triangle ABC´.
Rotate this triangle about A 40º to get triangle AB´C´´



Join C to C´´
Why is the triangle isosceles?
Calculate the angles of the triangle.

Adapting

Using an isosceles triangle with the same angles, reflect in AB and then rotate 20º.
Try a rotation of 120º.
Try different angles of rotation.

Extending

Try different isosceles triangles as the starting point.
Generalise – given the odd angle of the isosceles triangle (α) and the angle of rotation (β) predict the angles of the final triangle.

Mathematics

- reflection
- rotation
- properties of isosceles triangles
- angles in triangles

Other notes

For combinations such as angle ABC of 40º with a rotation of 100º you will not get a triangle, as CC´´ passes through A.

Further Challenges

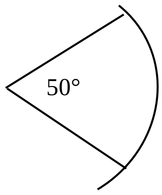
Section notes

Here are some extra tasks that offer different mathematics connected to the isosceles triangle. Enjoy!

FC1: From the Isosceles Triangle to ...

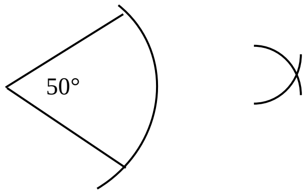
Task:

Towards one edge of your sheet, plot a point
Using a pair of compasses, draw an arc of radius 8cm, centre the point.

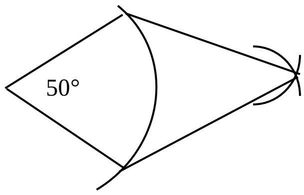


Draw two radii, so that the angle is 50°.

Use the points where the radii and arc intersect as centres and draw two arcs of the same radius, so that they intersect.



Join the points to form a quadrilateral. What sort of quadrilateral is it? Why?



Describe the properties of the quadrilateral.
Draw the two diagonals.
Why are the diagonals perpendicular to each other?
Where are the isosceles triangles?

Adapting

Repeat the task with different starting angles.
When do you get shapes that have two lines of symmetry?

Extending

Extend the original task by drawing pairs of arcs both sides of the first arc.
What two shapes do you get? How are they related?

Extend the original task by exploring how the areas of the two isosceles triangles relate to each other.
How is the ratio of their areas related to the arcs drawn?

Mathematics

- equality of radii and the equal sides of a shape
- lines of symmetry and the perpendicular bisector – why is the line connecting image points bisected at right angles by the line of symmetry?

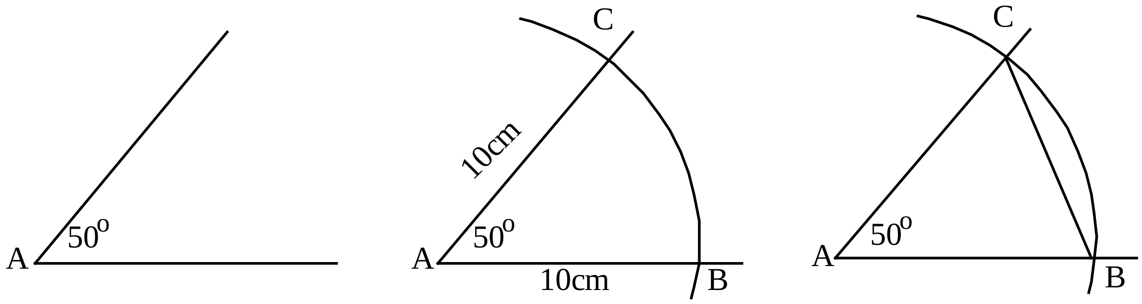
Other notes

This is the more usual construction for the angle bisector based on constructing two isosceles triangles and joining the line of symmetry.

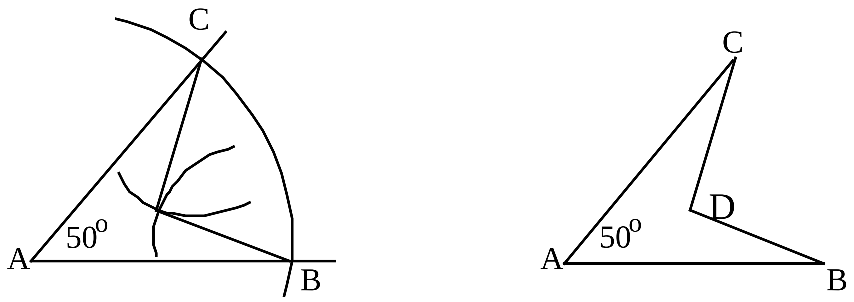
FC2: An Arrowhead with a Fixed Angle.

Task:

Use a large sheet of paper.
Draw an angle of 50°.
Label to corner A. Draw an arc of radius 10cm, centre A
Label where the arc intersects the line segment B and C.



Draw a small arc of a circle towards A, centre B, radius 6cm.
Draw a small arc of a circle towards A, centre C, radius 6cm.
Mark the point where the arcs cross – label the point D.
Join the points BDC.



Measure the angles. Are any angles equal? Why?
Find their sum.

Try other starting angles.

- What is the largest starting angle you can have?

Adapting

Find the area of the arrowheads.

Can the area of the arrowhead ever be less than half the area of the triangle ABC.

Extending

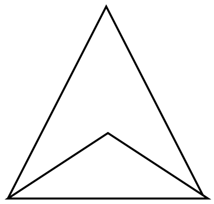
Construct the arrowhead inside a circle, by drawing the original triangle and constructing the circumcircle

Mathematics

- use of compasses
- properties of the isosceles triangle
- properties of compound shapes, including symmetry
- lines of symmetry as angle and perpendicular bisectors.

Other notes

Use isosceles triangles to explain why the shape has one line of symmetry and why you have a pair of equal angles.

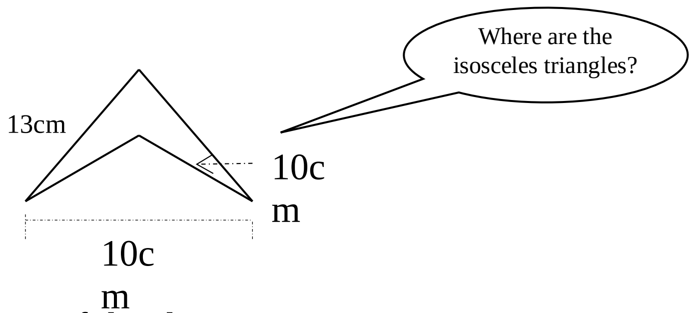


FC3: An Arrowhead with Fixed Diagonal.

Task

Use a large sheet of paper,.
Draw a line segment 10cm long.
Label the ends of the segment A and B.
Draw a small arc of a circle above AB, centre A, radius 13cm.
Draw a small arc of a circle above AB, centre B, radius 13cm.
Mark the point where the arcs cross – label the point C.
Join the points ACB.

Construct point D on the same diagram using arcs of radius 10cm, centres A and B.
Join ADB.



Describe the symmetries of the shape ACBD.

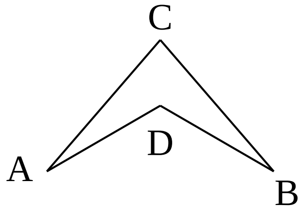
Adapting

Draw some other arrowheads of sides 13cm, 13cm, 10cm, 10cm.

Measure the angles.

Extending

Is it possible to have an arrowhead where angle ADB is twice the size of angle ACB?



Mathematics

- use of compasses
- properties of the isosceles triangle
- properties of compound shapes, including symmetry
- lines of symmetry as angle and perpendicular bisectors.

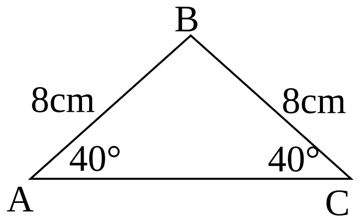
Other notes

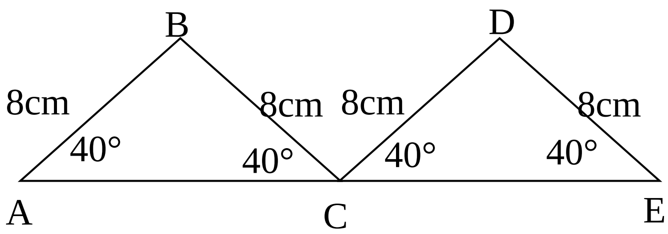
The arrow-head is a much ignored quadrilateral. The students may enjoy playing with this activity.

FC4: Isosceles Triangles in a Line.

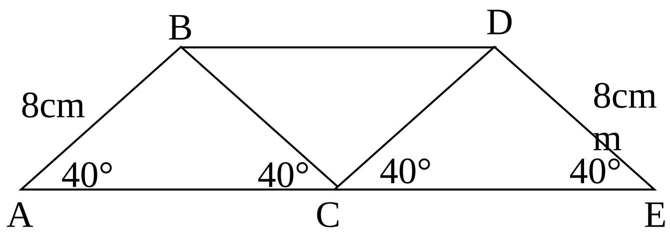
Task:

Draw an isosceles triangle with equal sides of 8cm and equal angles of 40° (ABC).
Continue the non-equal side (AC) and draw a congruent triangle on this line touching the original triangle at C.





Join the points BD to form another triangle.
Find all the angles in the figure.



Measure the lengths of BD and AE.
What shape is ABDE?
How would you convince someone of this?

Adapting

Draw the shape beginning with a different isosceles triangle.

Find the area of the triangles and the shape ABDE.

Construct the quadrilateral using a DGS.

Extending

Is the quadrilateral ABDE cyclic?
Construct another of the isosceles triangles on BD and work on similar triangles.
Draw another layer on triangles below AE. What can be deduced?

Mathematics

- parallel lines and alternate angles
- parallel lines and corresponding angles

Other notes

This activity gives a different approach to working on angles on parallel lines.