## Learning mathematics in adolescence

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## Learning algebra

## What most learners can do already that helps with algebra

* Learners naturally interpret symbolic and diagrammatic stimuli in terms of previous experience of similar forms
* Learners naturally generalise, look for patterns and habits, and familiar objects
* Generalisations and assumptions are usually connected to immediate visual impact or symbols, diagrams, graphs etc.
* Mental arithmetic strategies can provide a basis for understanding algebraic structures e.g. distributivity
* Devise methods of recording new situations
* Very young students can accept letters and symbols standing for numbers when they have quantitative relationships to express; they can use letters to stand for ‘hidden’ numbers and also for ‘any’ number if they understand what is being expressed
* Young students can construct formulae for themselves, at least in words if not symbols, if they have sufficient understanding of the relationships and operations and underlying situations
* Learners can connect, relate and compare different representations of the same phenomena if they have control of the technology
* Learners can work out relationships between number pairs if they have wide experience of operations, and this seems to help with expressing functions more generally
* Learners who deconstruct diagrams, relationships and situations can identify functional relationships better than those who only have access to pattern-generation
* Students learn from immediate feedback e.g. in ICT use
* Learners can construct symbolic instructions when using spreadsheets to explore functions
* Learners can distinguish between different uses of letters (variable, unknown, constant) if the situation has meaning for them

## What learners do naturally that hinders learning algebra

* want algebra to be just like arithmetic, because that is what they are familiar with and ‘letters stand for numbers’
* make intuitive assumptions and sensible, pragmatic reasoning about an unfamiliar notation system
* make analogies with symbol systems used in everyday life, in other parts of mathematics or in other school subjects
* try to apply what they have learnt most recently, whether relevant or not
* tend to fall into well-known habits and assumptions about using letters like numbers, e.g. *a*0 is ten times *a*
* make erroneous assumptions about layout and surface features without thinking about meaning
* assume that letters stand for numbers in some alphabetical way
* want to calculate and use arithmetical rules whenever possible
* interpret ‘=’ as an instruction to calculate
* tend to use number facts and guess-&-check rather than algebraic methods if possible
* their mental and written methods may not relate to symbolic notation
* persist in additive methods rather than using multiplicative and exponential methods
* tend to use their rules for reading and other false priorities when combining operations, i.e. interpreting left to right, doing addition first, using language to construct expressions, etc. They need to develop new priorities.
* hang on to familiar rules (e.g. stick with simplistic pattern-spotting); they do not immediately adopt new rules which contradict intuitive reactions
* see manipulatives, mnemonics, methods as ‘something else to learn’ unless they are connected to algebraic meanings through language and varied experiences
* focus on the most obvious behaviour, for example term-to-term in pattern-generation
* use methods they are confident about in new situations which are vaguely similar; this is a common source of error.
* confuse different metaphors, e.g. ‘balance’, ‘function machine’, ‘doing things to both sides’ ‘equivalence’.
* create their own action-based concrete understandings to substitute for abstract conceptualisations

### Recommendations for teaching algebra

* Algebra is the mathematical tool for working with generalities, and hence should permeate lessons so that it is used wherever mathematical meaning is expressed. Its use should be commonplace in lessons
* Teachers should avoid using published and web-based materials which exacerbate the difficulties by over-simplifying the transition from arithmetic to algebraic expression, mechanising algebraic transformation, and focusing on algebra as ‘arithmetic with letters’
* Shifting from arithmetical to algebraic understanding takes time, multiple experiences, and clarity of purpose
* Students need support in shifting to representations of generality, understanding relationships, and expressing these in conventional forms.
* Students have to shift focus from calculation, quantities, and answers to structures of operations and relations between quantities as variables. This shift takes time and multiple experiences.
* Students should have multiple experiences of constructing algebraic expressions for structural relations, so that algebra has the purpose of expressing generality.
* The role of ‘guess-the sequence-rule’ tasks in the algebra curriculum should be reviewed: it is mathematically incorrect to state that a finite number of numerical terms indicates a unique underlying generator.
* Students need multiple experiences over time to understand: the role of negative numbers and the negative sign; the role of division as inverse of multiplication and as the fundamental operation associated with rational numbers; and the meaning of equating algebraic expressions
* Substitution should be used purposefully for exemplifying the meaning of expressions and equations, not as an exercise in itself. Matching terms to structures is more useful than using them to practice substitution.
* ICT in the learner’s control, in the teacher’s control, and in shared control, can support the shifts of understanding that have to be made
* Teachers should encourage the use of symbolic manipulation, using ICT, as a set of tools to support transforming expressions for mathematical understanding.

## Learning new mathematical ideas

## How learners process new experiences

* Learners notice variation, obvious characteristics, and invariance in examples and experiences
* Learners can be guided to focus on critical aspects by controlled use of variation
* Learners relate new experiences to the most recent similar experience; it is common for visual and surface similarities to dominate
* Learners generalize from given examples, and this might lead to generalizing irrelevant features
* The form of representation, and how changes and variation is offered, is a critical influence on interpretation
* Automaticity can be helpful to get a sense of structure, but can also create obstacles to adaptation (e.g. when negative numbers or unfamiliar transpositions are introduced)
* If information is only presented as declarative knowledge, or as instructions, then learners are unlikely to develop conceptual understanding and adaptive reasoning

## What all students can do that can be drawn on by teachers

There is evidence to show that, with suitable environments, tools, images and encouragement, learners can and do use their general perceptual, comparative and reasoning powers in mathematics lessons to:

* generalise from what is offered and experienced
* look for analogies
* identify variables
* choose the most efficient variables, those with most connections
* see simultaneous variations
* understand change
* reason verbally before symbolising
* develop mental models and other imagery
* use past experience of successful and unsuccessful attempts
* accumulate knowledge of operations and situations to do all the above successfully

## What students can do with multiple experiences over time

* Students are more likely to identify quantities and relationships in situations, given extended experience of doing so
* Students can get better at thinking about and analysing mathematical situations
* Students who spend a significant time working on complex problems are not disadvantaged with respect to procedural work, and can often pick up procedures very quickly or devise them for themselves.
* Students can sort out earlier confusions if the tasks encourage them to discuss their confusions
* Students can sometimes do better if they are helped to use appropriate metacognitive strategies during their work
* Students can apply taught problem-solving heuristics, especially if they have to express them in writing or talk, and are prompted to do so while pursuing a task,
* Students may understand the modelling process better if they have to construct models of situations which then are used as models for new situations.
* There is no unique answer to the questions of when and why students can or cannot solve problems – it depends on the type of problem, the curriculum aim, the tools and resources, the experience, and what the teacher emphasizes.
* Students who can use available technology are better at problem solving and have more positive views of mathematics
* Students who work in computer-supported multiple representational contexts over time can understand and use graphs, variables, functions and the modelling process.

## What students do naturally which is not helpful but has to be taken into account

Learners tend to respond to new stimuli in the following ways:

* persistence of past methods, child methods, and application of procedures without meaning
* inappropriate application of recently-learned methods
* not being able to interpret symbols and other representations
* having limited views of mathematics from their past experience
* confusion between formal and contextual aspects
* inadequate past experience of a range of examples and meanings
* over-reliance on visual or linguistic cues, and on application of procedures
* persistent assumptions about addition, more-more/same-same, linearity, and confusions about quantities.

**Recommendations for teaching**

* Take into account students’ natural ways of dealing with new perceptual and verbal information (see summaries above), including those ways that are helpful for new mathematical ideas and those that obstruct their learning
* Allow enough time for students to adapt to new meanings and move on from earlier methods and conceptualisations; they should give time for new experiences and mathematical ways of working to become familiar in several representations and contexts before moving on.
* Choice of tasks and examples should be purposeful, and they should be constructed to help students shift towards understanding new variations, relations and properties. Such guidance includes thinking about learners’ initial perceptions of the mathematics and the examples offered. Students can be guided to focus on critical aspects by the use of controlled variation, sorting and matching tasks, and multiple representations.
* Students should be helped to balance the need for fluency with the need to work with meaning.
* Allow for students to have multiple experiences, with multiple representations, over time to develop mathematically appropriate ‘habits of mind’.
* The learning aims and purpose of tasks should be clear: whether they are: to develop a broader mathematical repertoire; to learn modelling and problem-solving skills; to understand the issues within the context better etc.
* Students need help and experience to know when to apply formal, informal or situated methods
* Students need a repertoire of appropriate functions, operations, representations and mathematical methods in order to become good applied mathematicians. This can be gained through multiple complex experiences over time rather than single complex, or multiple simple, experiences.