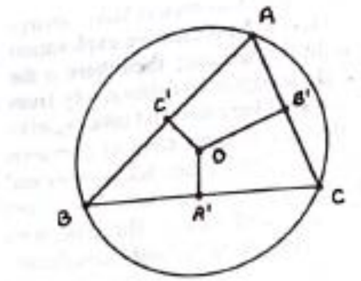
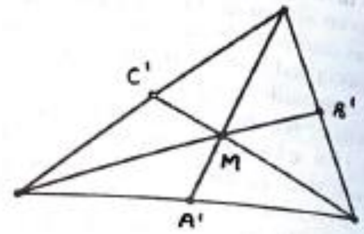
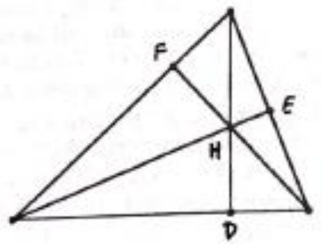
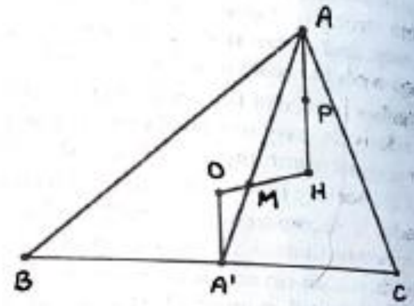
Some Triangle Geometry

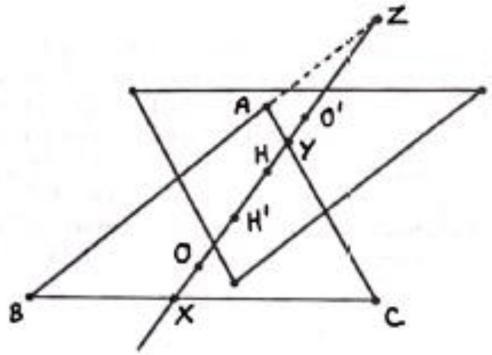
Dick Tahta

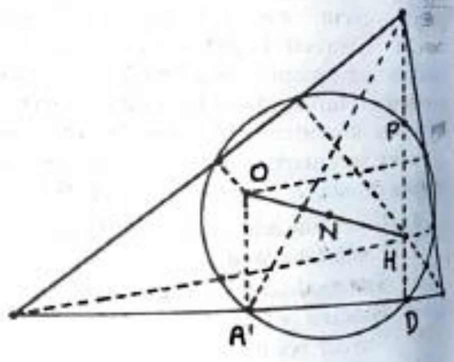












1: Euler line and medial circle

• The perpendicular bisectors of the sides of a triangle are concurrent at a point which is equidistant from the vertices.

The perpendicular bisectors are also called mediators; they concur at a point O, called the circumcentre, which is the centre of the circumcircle through the vertices of the triangle.

• The lines joining vertices to midpoints of opposite sides are concurrent at a point that trisects the lines.

The lines are called medians and they concur at a point M called the median point. The median point is the centre of gravity of equal weights at the vertices and is often also called the centroid.

• The perpendiculars from the vertices of a triangle to the opposite sides are concurrent, at a point which is isogonal to the circumcentre.

The perpendiculars are called altitudes; they concur at a point H called the orthocentre.

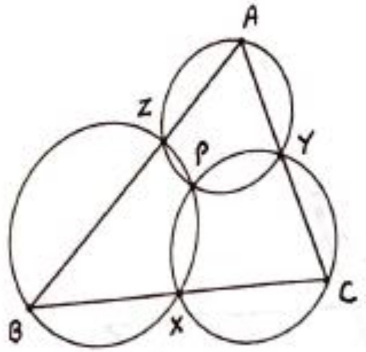
• The circumcentre O, the median point M, and the orthocentre H, are collinear, with M trisecting OH.

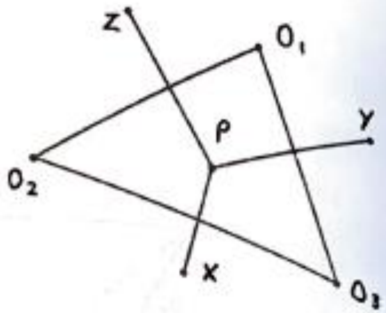
The line OMH is called the Euler line.

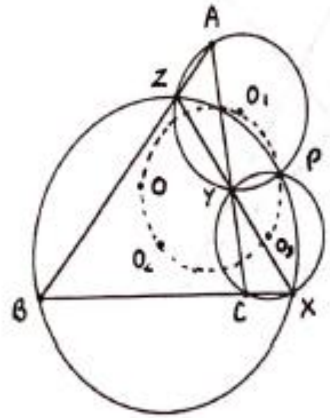
• The Euler line meets the sides of the triangle in points X, Y, Z: the Euler lines of triangles AYZ, BZX, CXY form a triangle congruent to ABC and with the same Euler line.

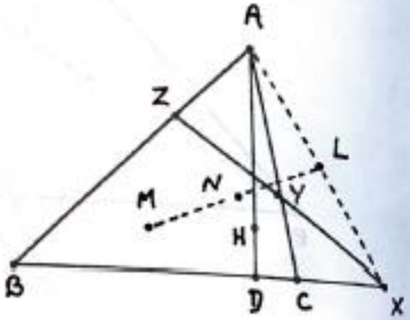
• The midpoint of OH is the centre of the circle through the midpoints of the sides; this circle also passes through the feet of the altitudes and the midpoints of the lines joining H to the vertices.

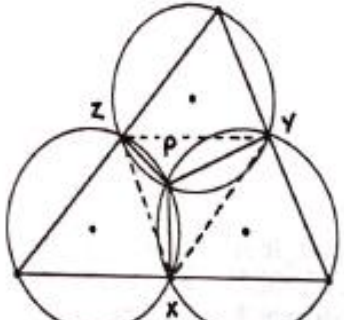
The circle through the midpoints of the sides is called the medial circle, or the nine-points circle after the nine points it passes through, as described in the previous paragraph.

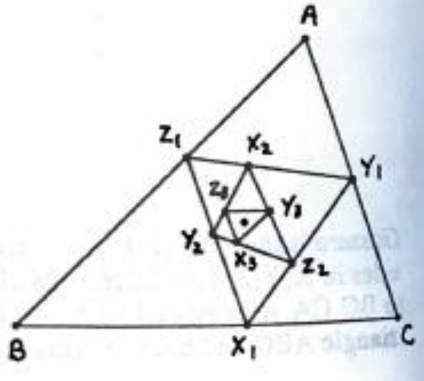












2: Miquel circles

• With any three points X, Y, Z on the sides of the triangle ABC, the circles AYZ, BZX, CXY meet in a point.

The circle are called the Miquel circles for the triple X, Y, Z, and the point common to them is called the Miquel point.

• The centres of a set of Miquel circles form a triangle that is similar to the original triangle.

• The Miquel point of a collinear triple of points lies on the circumcircle.

• The centres of the Miquel circles of a collinear triple and the circumcentre of the triangle lie on a further circle also passing through the Miqel point.

The previous result can be restated as follows: four lines form four triangles whose circumcircles meet at a point, and the circumcentres lie on the a circle also passing through the point. This can then be generalised: five lines form ten triangles whose circumcentres meet in fours at five points which lie on a circle.

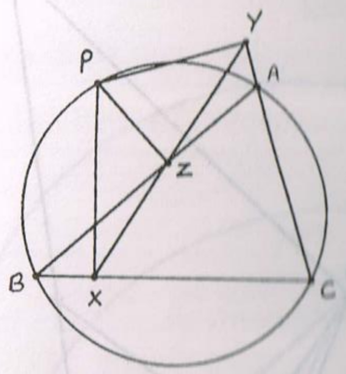
• The circumcentres of the four triangles formed by four lines are concyclic, and the orthocentres are collinear.

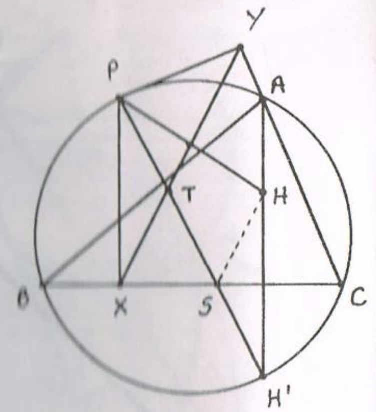
• The midpoints of the diagonals of a quadrilateral are collinear.

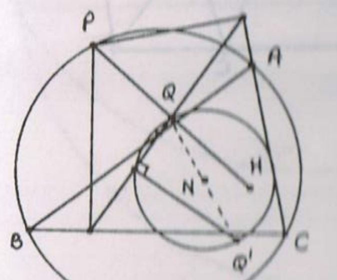
• Any point P is the Miquel point of the triple formed by the feet of the perpendiculars from a point to the sides of a triangle; the Miquel circles in this case have diameters PA, PB, PC.

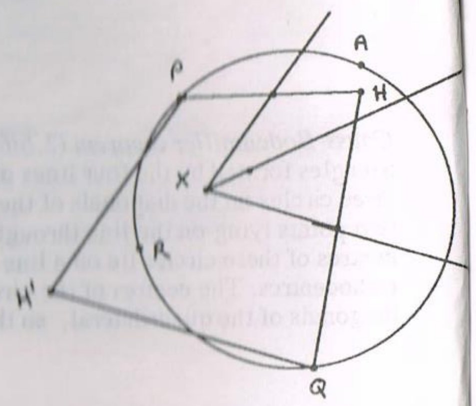
The triangle formed by the triple is known as the pedal triangle of the point P with regard to the triangle ABC. For example, the medial circle is the circumcircle of the pedal triangle of the orthocentre as well as the pedal triangle of the circumcentre.

• The pedal triangle of the pedal triangle of the pedal triangle of a point is similar to the original triangle.









3: Pedal (Simson) lines

- The feet of the perpendiculars from a point P on the circumcircle are collinear, and conversely,

  In this case, the pedal triangle of P collapses into a straight line called the pedal line of P with regard to the triangle ABC. The pedal line is also often called the Simson line, through an erroneous ascription, and sometimes the Wallace line after the actual discoverer.

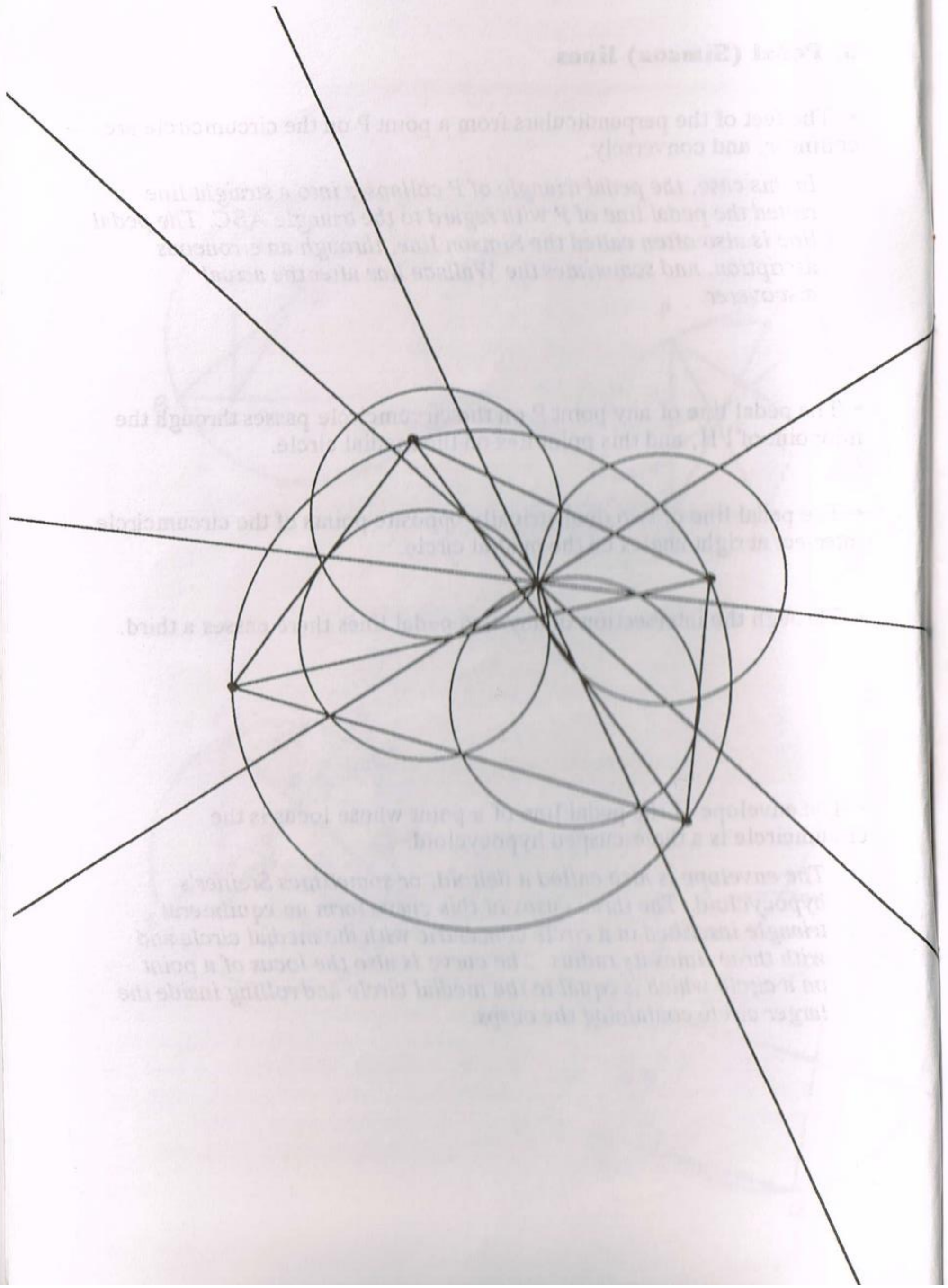
- The pedal line of any point P on the circumcircle passes through the midpoint of PH, and this point lies on the medial circle.

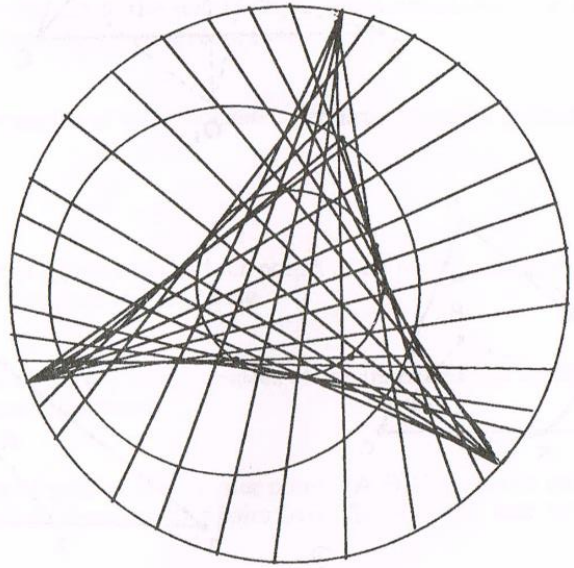
- The pedal line of two diametrically opposite points of the circumcircle intersect at right angles on the medial circle.

- Through the intersection of any two pedal lines there passes a third.

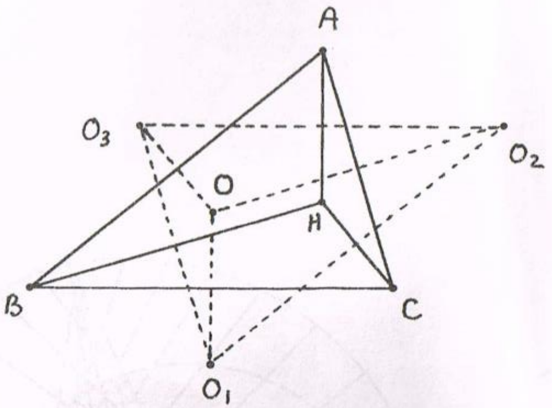
- The envelope of the pedal line of a point whose locus is the circumcircle is a three-cusped hypocycloid.

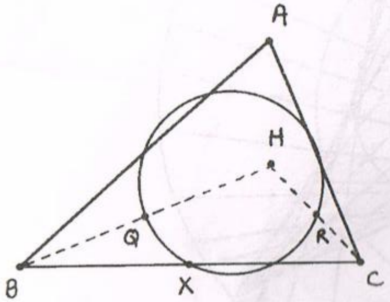
  The envelope is also called a deltoid, or sometimes Steiner's hypocycloid. The three cusps of this curve form an equilateral triangle inscribed in a circle concentric with the medial circle and with three times its radius. The curve is also the locus of a point on a circle which is equal to the medial circle and rolling inside the larger circle containing the cusps.

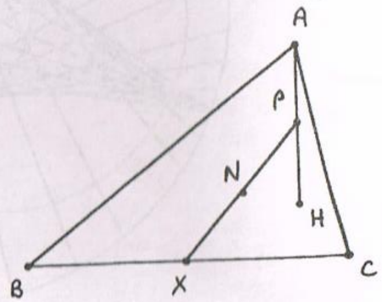


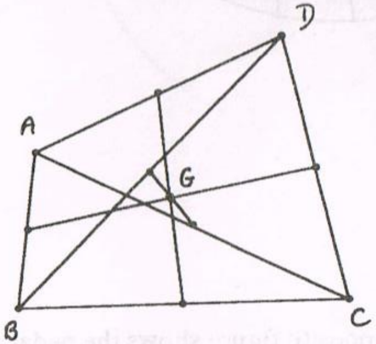


Some pedal line properties : the opposite figure shows the pedal lines of each of four concyclic points with regard to the triangle formed by the other three - these are collinear at the common point of the four medial circles. The above figure shows the envelope (Steiner's hypocycloid) of the pedal line of a point on the circumcircle.









4: Orthocentric sets

- Each vertex of a triangle is the orthocentre of the triangle formed by H and the other two vertices.

    A set of four points each the orthocentre of the other three is known as an orthocentric set.

- The circumcentres of the four triangles of an orthocentric set form another orthocentric set congruent to the first.

- The four triangles of an orthocentric set have the same medial circle.

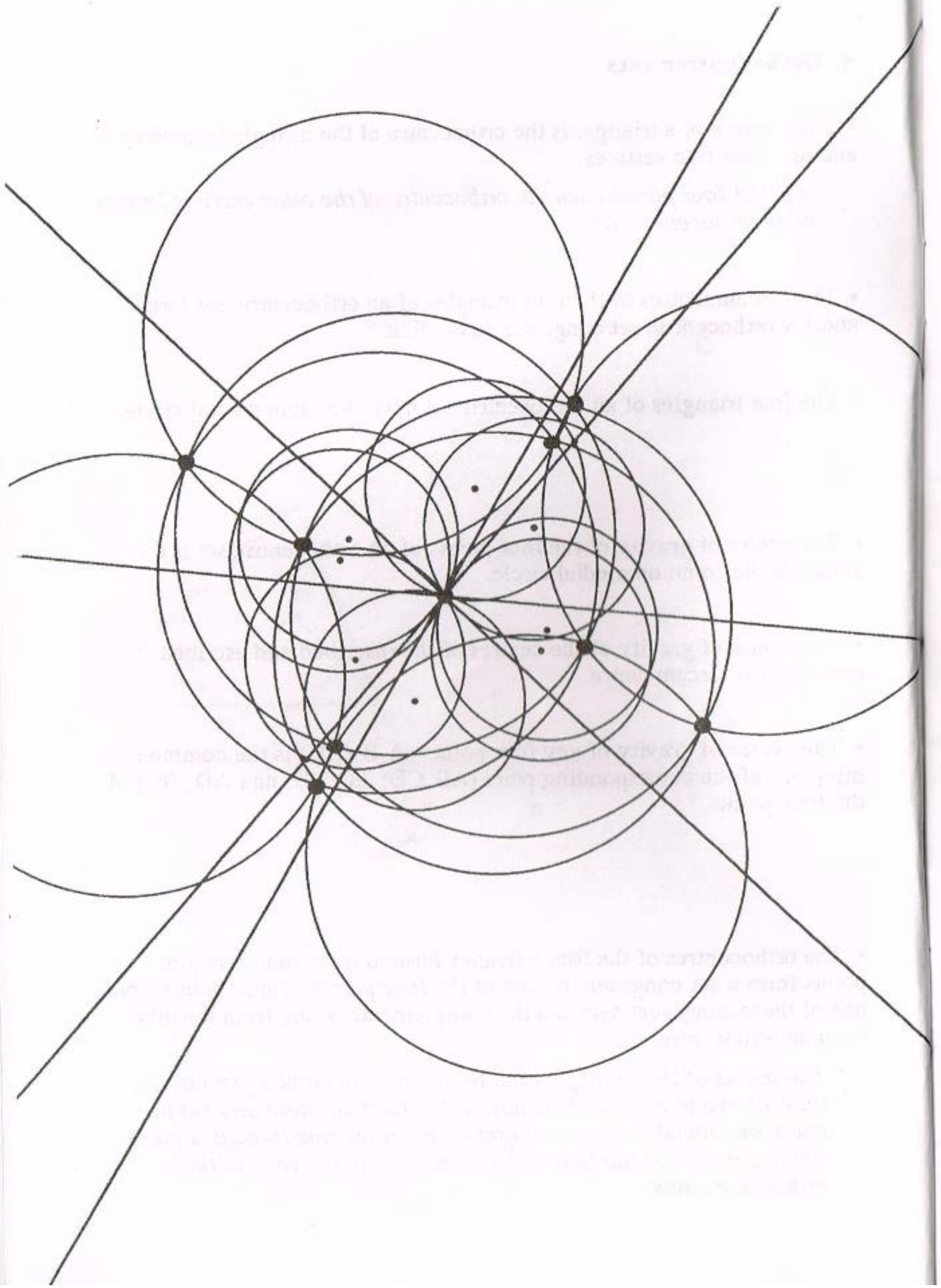
- The centre of gravity of the four points of an orthocentric set is the centre of the common medial circle.

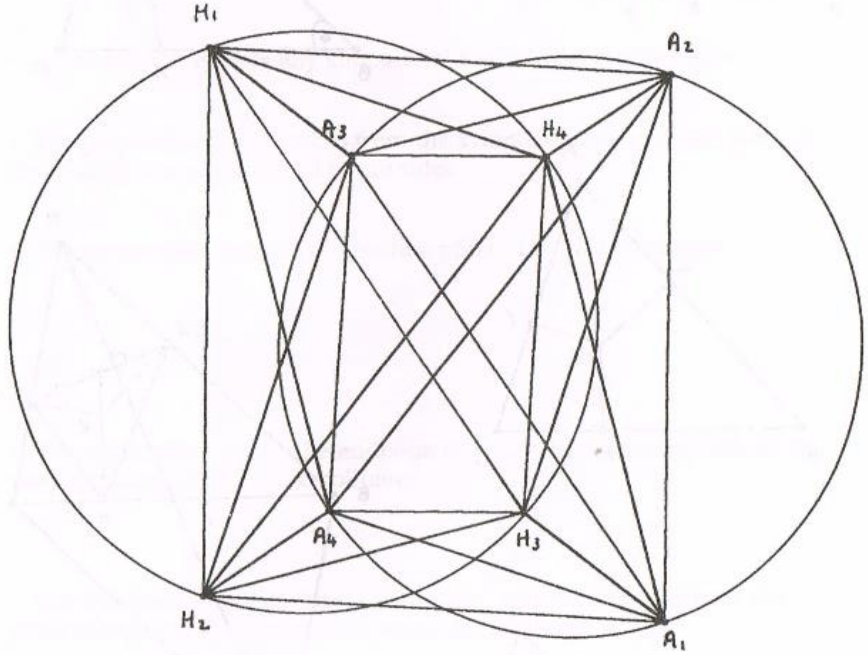
- The centre of gravity of the centres of the inscribed and escribed circles is the circumcentre.

- The centre of gravity of any four points (A,B,C,D) is the common midpoint of the corresponding pairs (AB,CD; AC,BD; and AD, BC) of the four points.

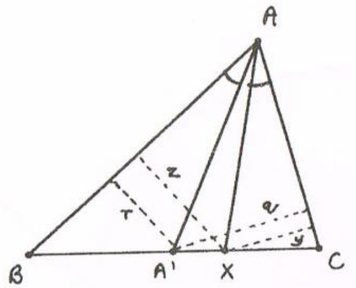
- The orthocentres of the four triangles formed from four concylic points form a set congruent to that of the four points. Three points from one of these congruent sets and the complementary one from the other form an orthocentric set.

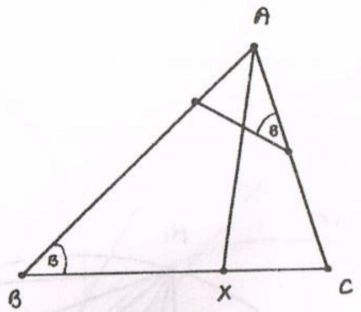
    The points of this configuration lie in fours on eight equal circles; they also form eight orthocentric sets. Each of these sets has a common medial circle : the eight such circles pass through a point.. This point lies on the four pedal lines common to each of the orthocentric sets.

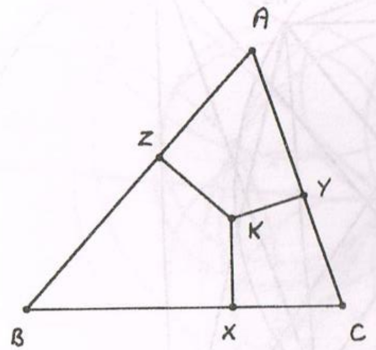


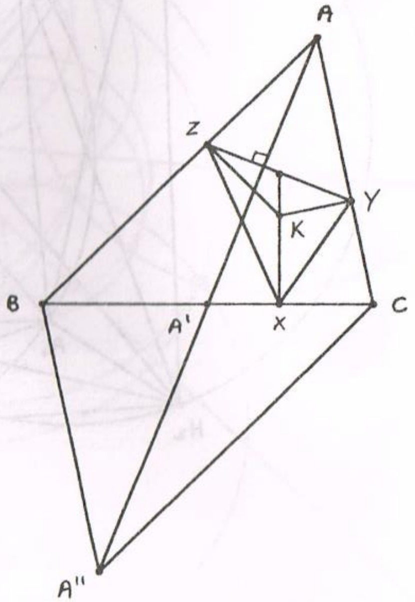


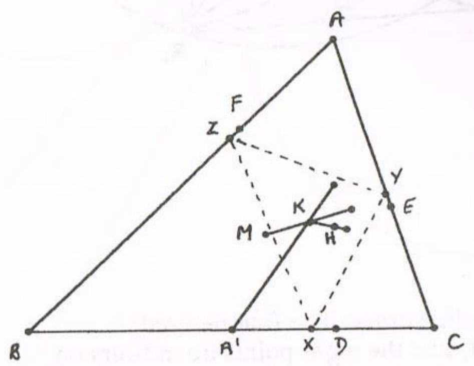
A configuration from four concyclic points : The four derived orthocentres form a congruent set; and the eight points lie in fours on eight equal circles with the various further properties given in 4.7.











5: Symmedian (Lemoine) point

• The isogonal conjugate of the median through A meets BC in X where BX/CX=c^2/b^2, etc, and AX, BY, CZ are concurrent.

The isogonal conjugates of the medians are called symmedians and they concur at a point K called the symmedian point, or sometimes the Lemoine point.

• A symmedian bisects any antiparallel to its corresponding side.

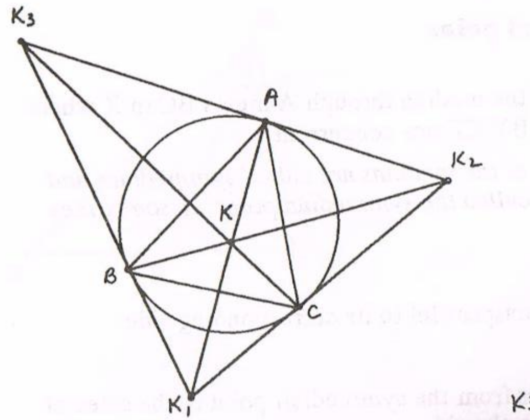
• The perpendicular distances from the symmedian point to the sides of the triangle are proportional to the sides.

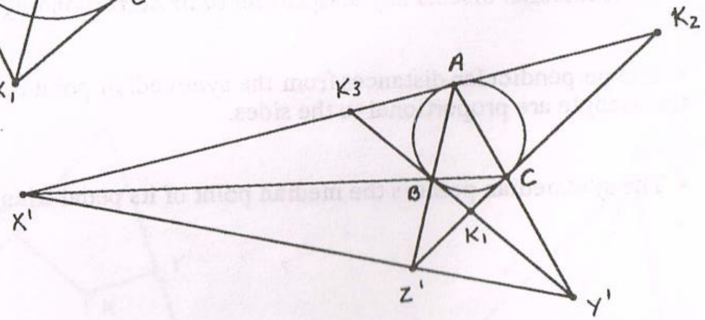
• The symmedian point is the median point of its pedal triangle.

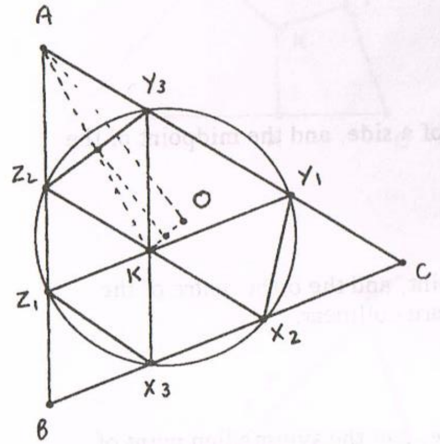
• The symmedian point, the midpoint of a side, and the midpoint of the corresponding altitude, are collinear.

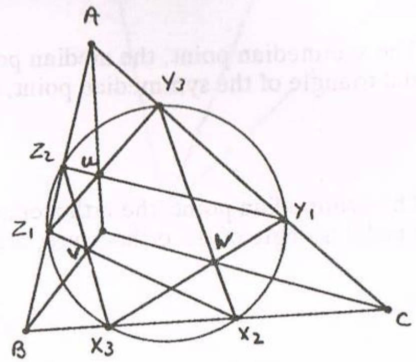
• The symmedian point, the median point, and the orthocentre of the pedal triangle of the symmedian point, are collinear.

• The symmedian point, the orthocentre, and the symmedian point of the pedal triangle of the orthocentre, are collinear.









6: Lemoine line and circle

- The tangents to the circumcircle at A,B,C meet at K_1,K_2,K_3: the lines AK_1,BK_2,CK_3 are concurrent at the symmedian point.

  The tangents are sometimes called exsymmedians, and their intersections are the exsymmedian points K_1, K_2, K_3.

- The exsymmedians meet the opposite sides in X', Y', Z' respectively: these points are collinear, lying on the polar of the symmedian point with regard to the circumcircle.

  The polar of the symmedian (Lemoine) point is called the Lemoine line.

- The lines through the symmedian point parallel to the sides of the triangle meet the sides in six points that lie on a circle,

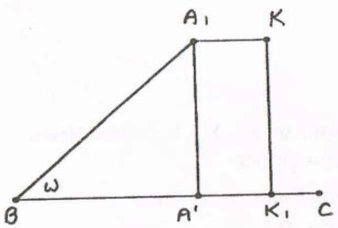
  The circle containing the six points is called the Lemoine circle.

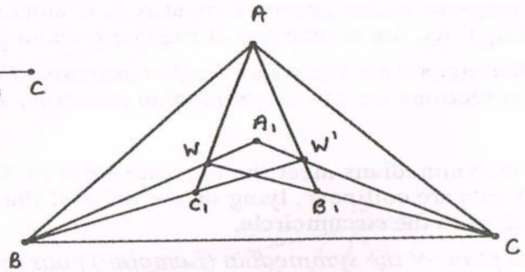
- The centre of the Lemoine circle is the midpoint of the symmedian point and the circumcentre.

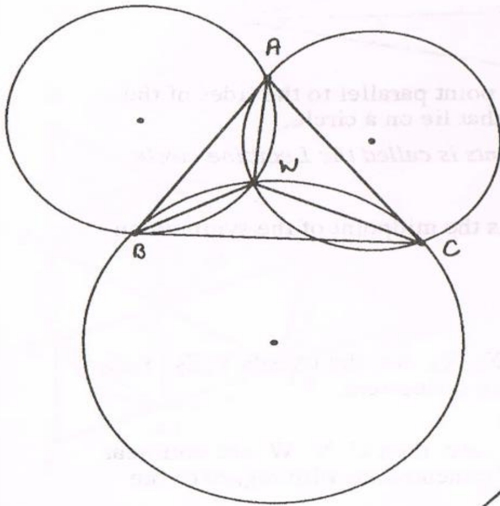
- The Lemoine circle meets BC at X_2,X_3, etc; the chords Y_1Z_2, Y_3Z_1 meet at U, etc, : then AU,BV,CW are concurrent.

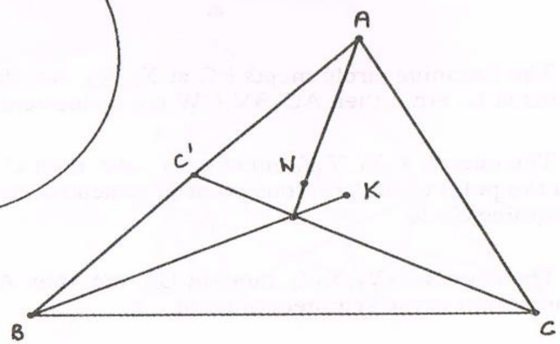
- The chords Y_1Z_1, Y_3Z_2 meet at U', etc: then U',V',W' are collinear on the polar of the previous point of concurrence with regard to the Lemoine circle.

- The chords X_2Y_1,X_3Z_1 meet at U", etc: then AU",BV",CW" are concurrent at the symmedian point.









7: Brocard points

- The line through the symmedian point parallel to BC meets the mediator of BC at A_1, etc: the angles A_1BC, B_1CA, C_1AB and A_1CB, B_1AC, C_1BA are equal.

  The common angle is called the Brocard angle.

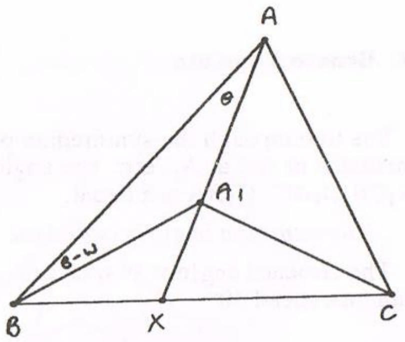
- The Brocard angle w is such that $\cot w = \cot A + \cot B + \cot C$; and w cannot exceed 30°.

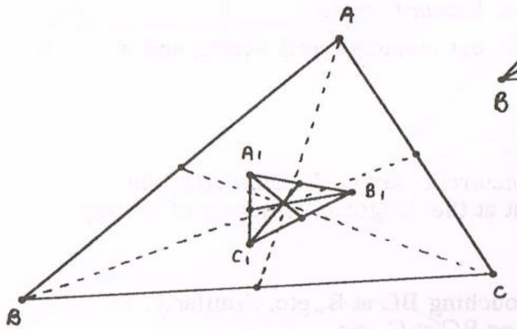
- The lines AC_1, BA_1, CB_1 are concurrent, say at W. Similarly, the lines AB_1, BC_1, CA_1 are concurrent at the isogonal conjugate of W, say W'.

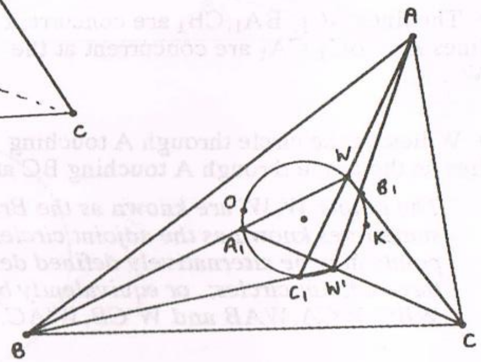
- W lies on the circle through A touching BC at B, etc. Similarly, W' lies on the circle through A touching BC at C, etc

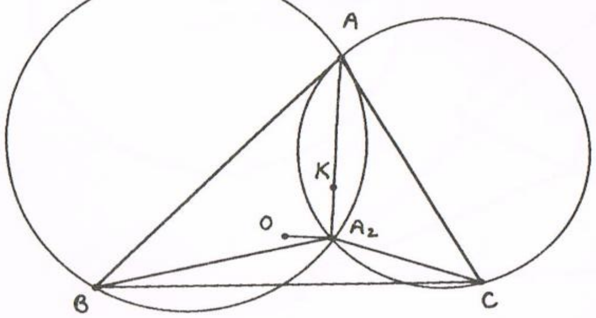
  The points W, W' are known as the Brocard points. The circles are sometimes known as the adjoint circles of the triangle. The Brocard points may be alternatively defined defined as the common points of three adjoint circles;, or equivalently by the property that the angles WBC, WCA, WAB and W'CB, W'AC, W'BA, are equal.

- The line AW, the symmedian through B, and the median through C, are concurrent; and similarly for the corresponding vertices.









8: Brocard triangle and circle

• The triangle $A_1B_1C_1$ is similar to the triangle ABC; moreover, AA_1BB_1,CC_1 are concurrent, so that the triangles are in perspective.

The triangle $A_1B_1C_1$ is known as the Brocard triangle, and its circumcircle as the Brocard circle.

• The median point of the Brocard triamgle is the median point of the orginal triangle ABC.

• The Brocard circle passes through the points W,W', the circumcentre O, and the symmedian point K.

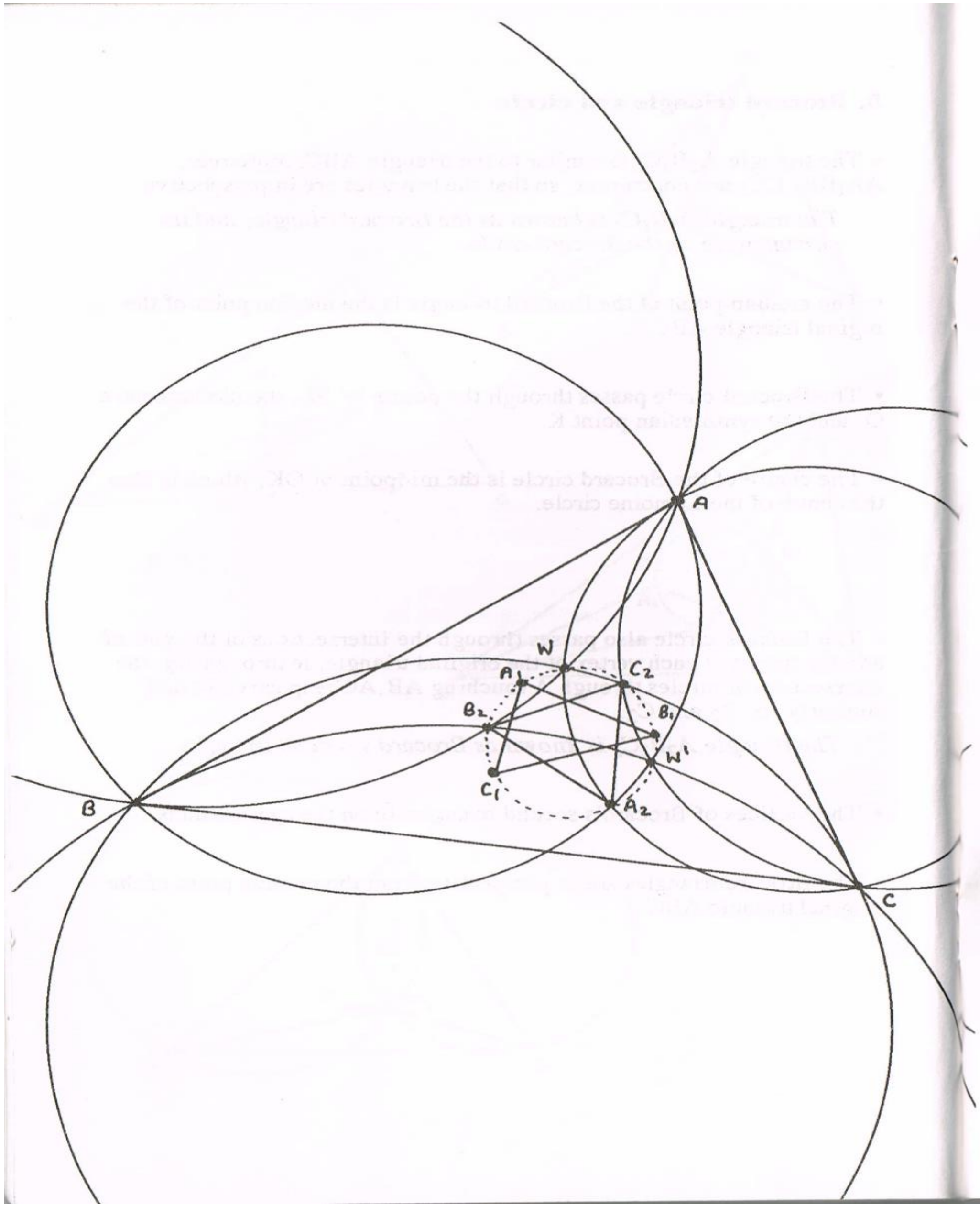
• The centre of the Brocard circle is the midpoint of OK, which is also the centre of the Lemoine circle.

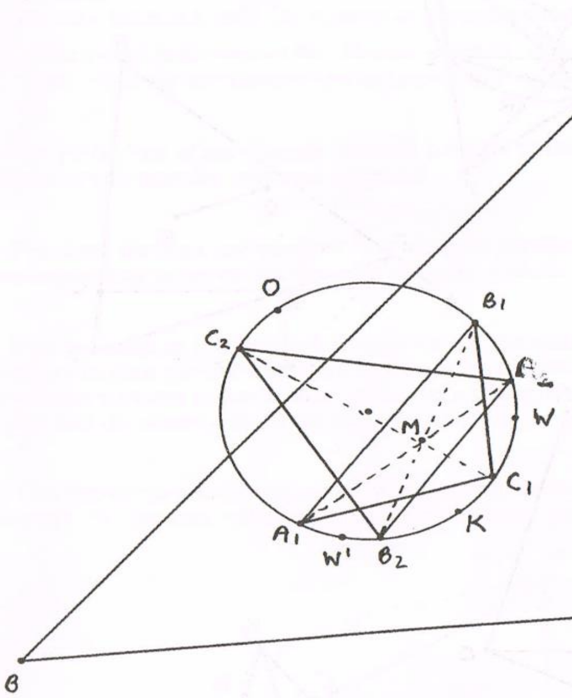
• The Brocard circle also passes through the intersections of the pair of adjoint circles at each vertex of the original triangle; ie through A_2, the intersection of circles through A touching AB,AC respectively, and similarly for B_2 and C_2.

The triangle $A_2B_2C_2$ is known as Brocard's second triangle.

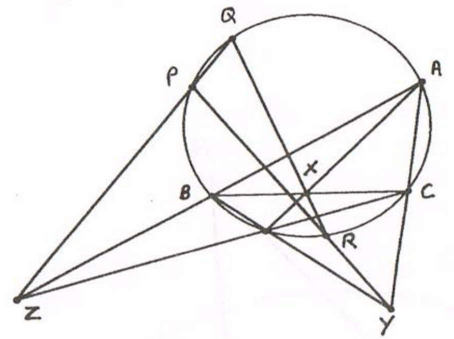
• The vertices of Brocard's second triangle lie on the symmedians.

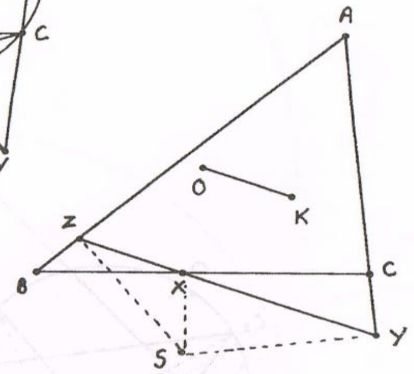
• The Brocard triangles are in perspective from the median point of the original triangle ABC.

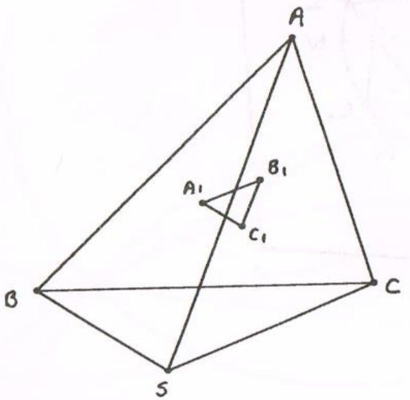


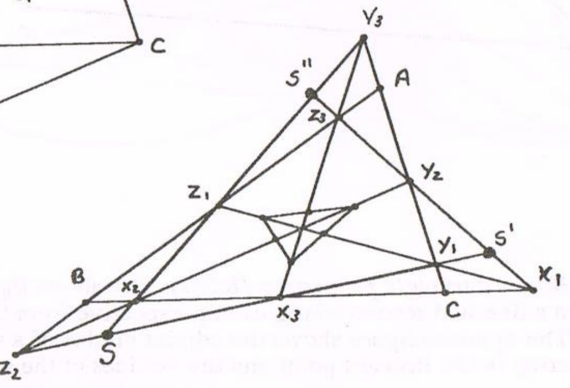


Some Brocard triangle properties (8.5/7): The above figure shows Brocard's first and second triangles in perspective from the median point. The opposite figure shows the adjoint circles of a triangle intersecting in the Brocard point and the vertices of the second Brocard triangle.









9: Steiner point

- The image of A in the mediator of BC is P, etc, and QR meets BC in X, etc: the lines AX, BY, CZ concur at a point on the circumcircle.

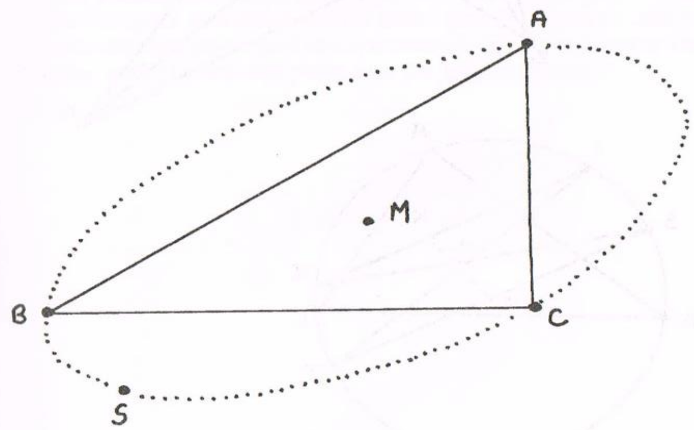
  This point is known as the Steiner point S. It has a number of properties (or alternative definitions) such as those listed below.

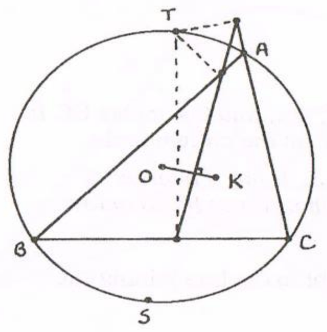
- The pedal line of the Steiner point is parallel to the line joining the circumcentre and the symmedian point.

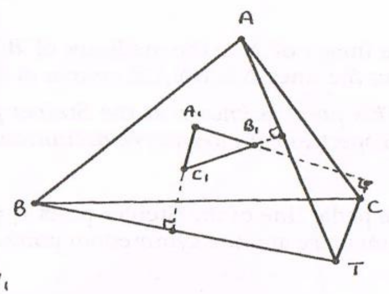
- The lines through the vertices of a triangle parallel to the corresponding sides of the Brocard triangle concur at the Steiner point .

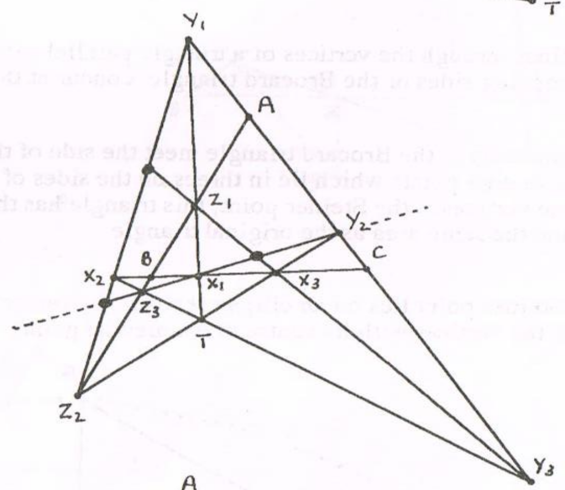
- The medians of the Brocard triangle meet the side of the original triangle in nine points which lie in threes on the sides of a triangle, one of whose vertices is the Steiner point; this triangle has the same median point and the same area as the original triangle.

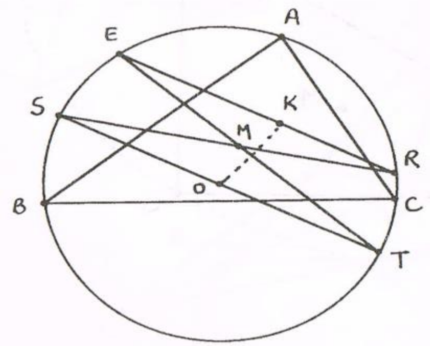
- The Steiner point lies on an ellipse (*known as Steiner's circumellipse*) through the vertices with its centre at the median point.











10: Tarry point

• The pedal line of the point on the circumcircle diametrically opposite the Steiner point is perpendicular to the line joining the circumcentre and the symmedian point.

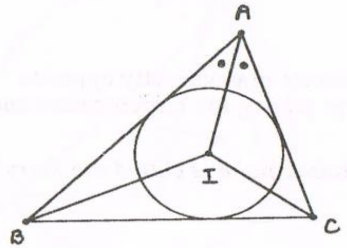
The point diametrically opposite the Steiner point is called the Tarry point.

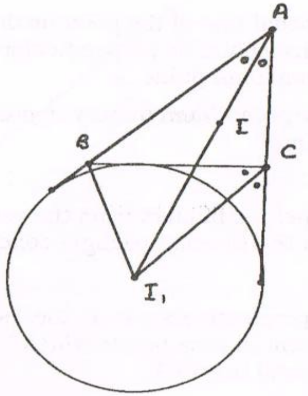
• The perpendiculars from the vertices of a triangle to the corresponding sides of the Brocard triangle concur at the Tarry point.

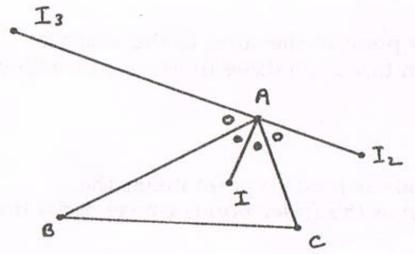
• The perpendiculars from the Tarry point to the sides of the triangle meet them in nine points which lie in threes on three lines, one of which is the pedal line of T.

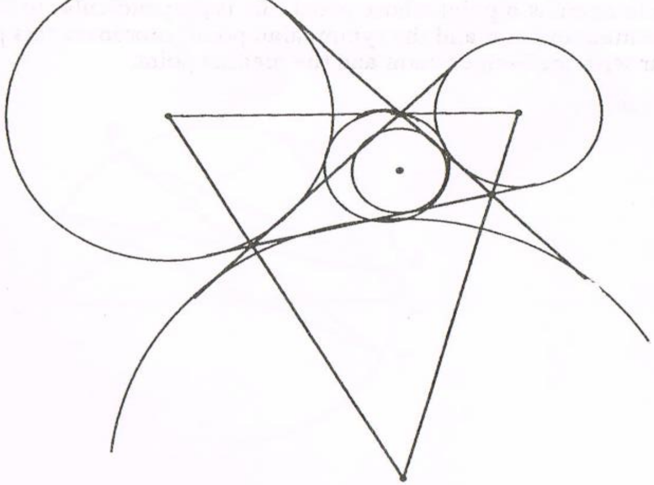
• The line joining the Tarry point and the median point meets the circumcircle again at a point (*known as the Euler point*) whose pedal line is parallel to the Euler line.

• The line joining the Euler point and the symmedian point meets the circumcircle again at a point whose pedal line is perpendicular to the line joining the median point and the symmedian point; moreover this point is collinear with the Steiner point and the median point.









11: Tritangent circles

• The internal angle-bisectors of a triangle are concurrent at a point which is the centre of the inscribed circle touching the sides.

• An internal angle-bisector of a triangle and the two external angle-bisectors at the other two vertices are concurrent at a point which is the centre of an escribed circle touching the sides.

The inscribed circle is called the in-circle, an escribed circle is called an ex-circle; the corresponding centres are the in-centre I, and the three ex-centres I_1, I_2, I_3. *The incircle and excircles are sometimes known as the tritangent circles.*

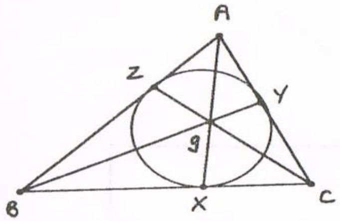
• The tritangent circles touch the medial circle.

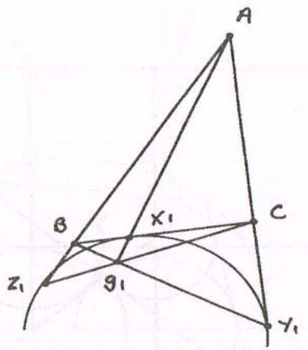
• The centres of the tritangent circles of a triangle form an orthocentric set.

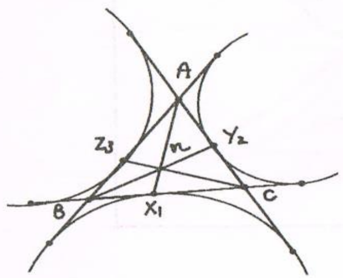
• The sixteen tritangent circles of the four triangles formed from an orthocentric set touch the common medial circle of the four triangles.

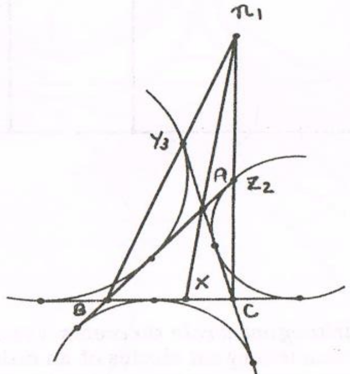
• The centres of the four in-circles of the triangles determined by four concyclic points form a rectangle.

An equivalent property is that the sum of the radii of the two in-circles for one triangulation of a cyclic quadrilateral is the same as that of the other triangulation. More generally, it then follows that the sum of the radii of the incircles is the same for every triangulation of a particular cyclic polygon. This property was known to Japanese mathematicians in the eighteenth century.









12: Gergonne and Nagel points

- The points of contact of the sides of a triangle with the in-circle are X, Y, Z: the lines AX, BY, CZ are concurrent.

- The points of contact of the sides of the triangle with the ex-circle opposite A are X_1, Y_1, Z_1, etc: the lines AX_1, BY_1, CZ_1 are concurrent, etc.

  The internal point of concurrence g is known as the Gergonne point; the other three points of concurrence, g_1, g_2, g_3 are called associated Gergonne points.

- The Gergonne point g is the symmedian point of the triangle XYZ; and the associated point g_1 is the symmedian point of the triangle X_1, Y_1, Z_1, etc.

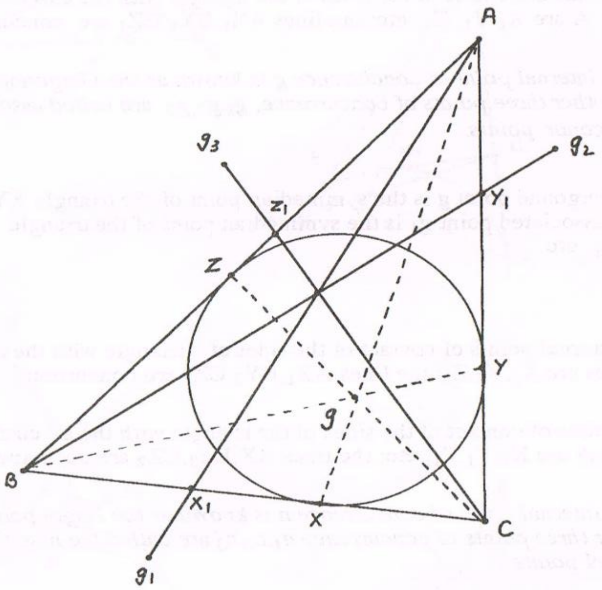
- The internal points of contact of the sides of a triangle with the three ex-circles are X_1, Y_2, Z_3: the lines AX_1, BY_2, CZ_3 are concurrent.

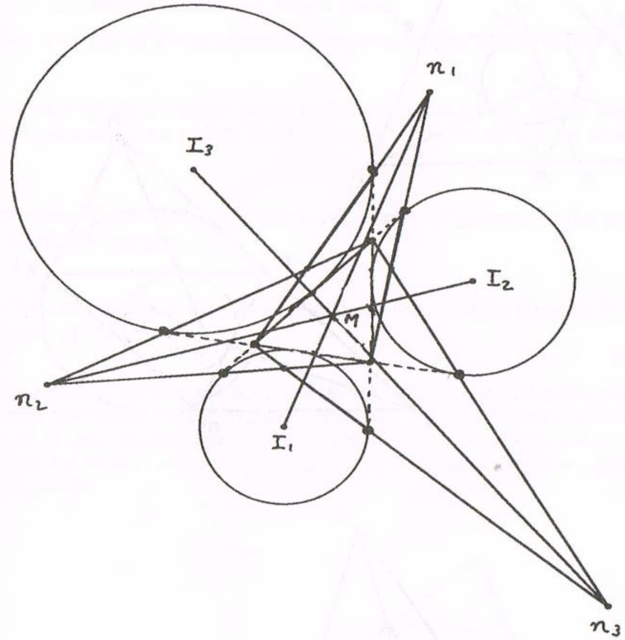
- The points of contact of the sides of the triangle with the ex-circle opposite A are X_1, Y_1, Z_1, etc: the lines AX, BY_3, CZ_2 are concurrent, etc.

  The internal point of concurrence n is known as the Nagel point; the other three points of concurrence n_1, n_2, n_3 are called the associated Nagel points.

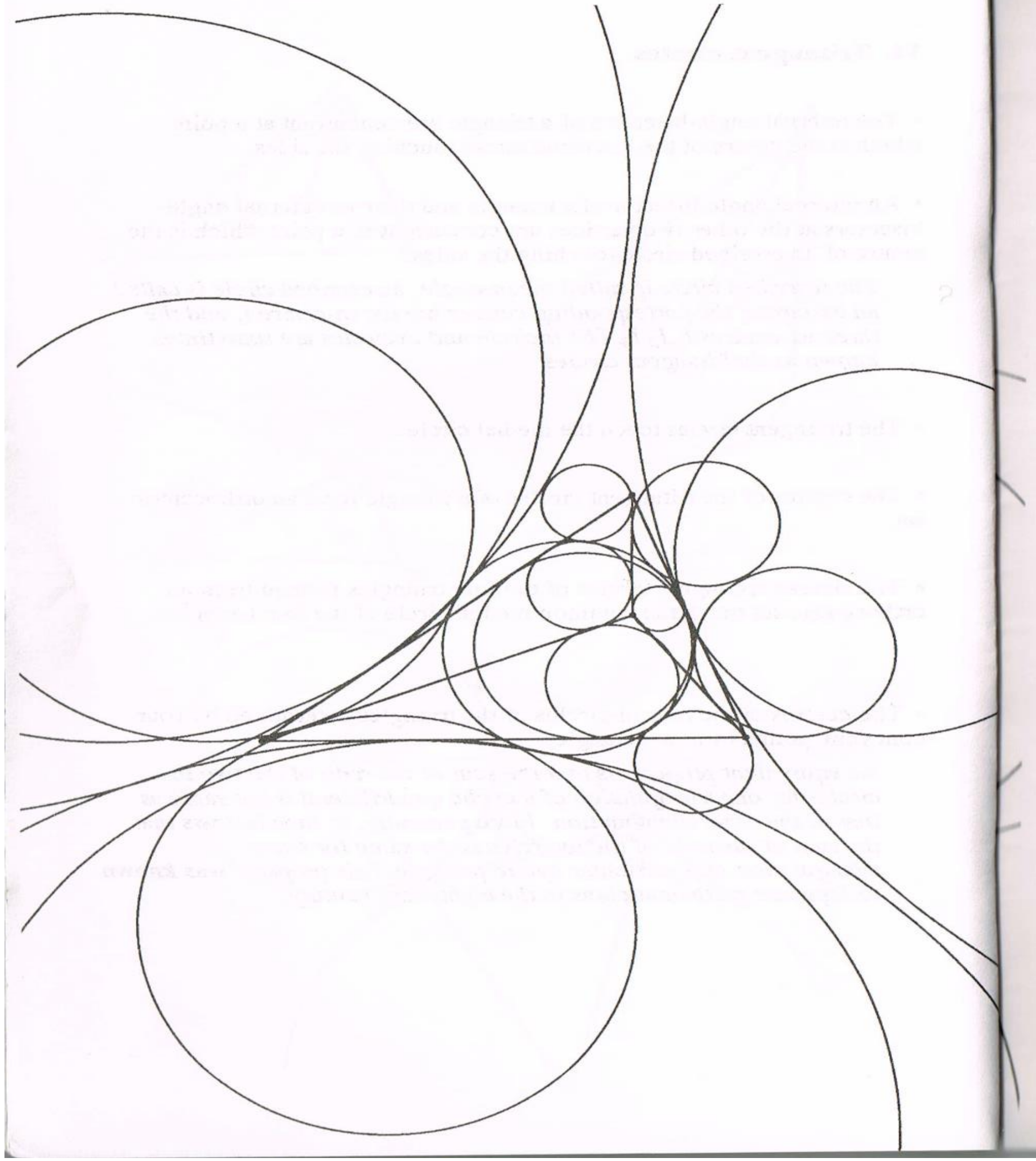
- Each Nagel point is the isotomic conjugate of its corresponding Gergonne point.

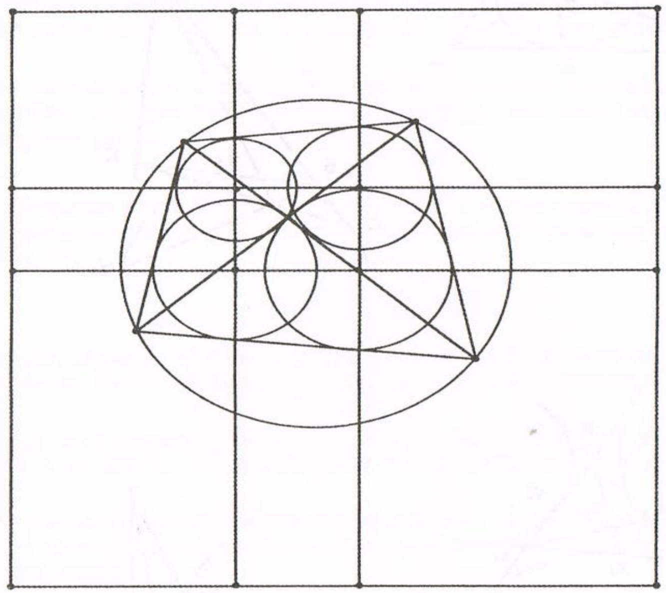
- The Nagel point, the in-centre, and the median point are collinear, with the median point trisecting the line joining the other two; and similarly for the associated Nagel points and their corresponding ex-centre.



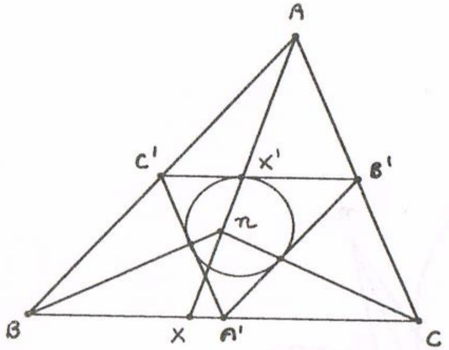


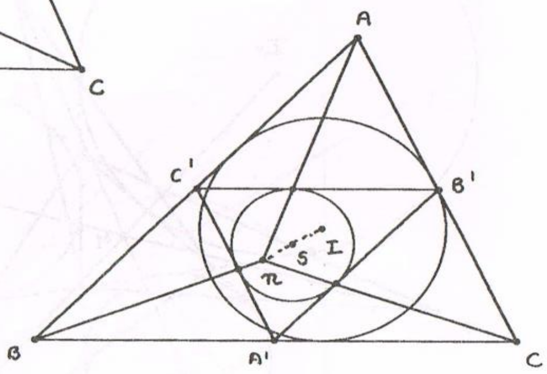
Some incidence properties of Gergonne and Nagel points .

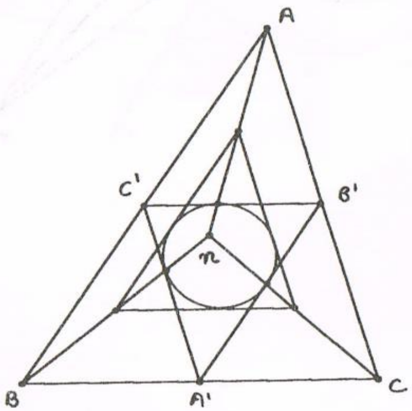




Some tritangent circle theorems (11.5/6): The opposite figure shows the sixteen tritangent circles of an orthocentric set. The above figure shows the rectangles formed by the sixteen centres of the tritangent circles of a concyclic set of four points.







13: Spieker circle

• The midpoints of the sides of the triangle ABC are A',B',C', and the incircle of the midpoints triangle touches B'C' at X', etc: the lines AX', BY', CZ' are concurrent at the Nagel point.

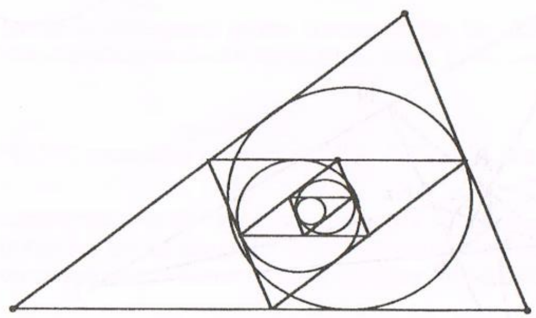
The in-circle of the midpoints triangle is called the Spieker circle.

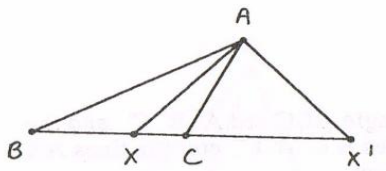
• The centre of the Spieker circle is the midpoint of the Nagel point n and the in-centre I; its radius is half that of the in-circle of the triangle ABC.

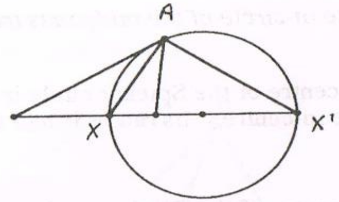
• The centre of the Spieker circle is the centre of gravity of the perimeter of the original triangle.

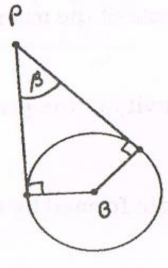
• The Spieker circle is also the in-circle of the triangle formed by the midpoints of An,Bn,Cn.

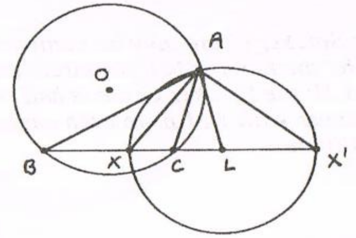
The Spieker circle may be compared with the medial (nine-point) circle: the latter is half the circumcircle, with its centre collinear with O,M,H; the Spieker circle is half the in-circle, with its centre collinear with I,M,n. In both cases, the median point is a centre of similitude.

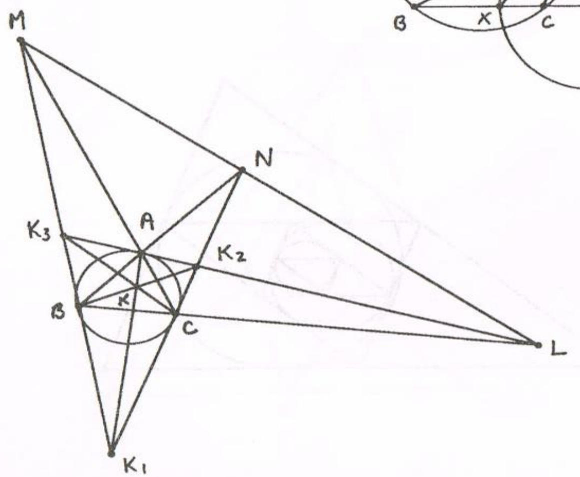












14: Apollonius circles

• The two angle bisectors at A meet the opposite side BC at X,X': these points divide BC 'internally and externally' in the ratio AB/AC.

This is Euclid, Book 6, proposition 3.

• The circle on XX' as diameter passes through A. This circle is the locus of a point such that PB/PC is constant - namely equal to AB/AC.

This property was given by Apollonius but it was already known to previous Greek mathematicians. The circle is known as the Apollonius circle (of the triangle) through A. There are corresponding circles through B and C.

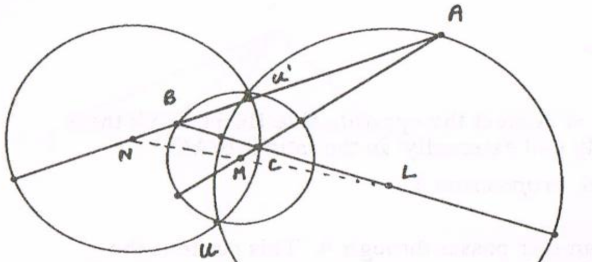
• The Apollonius circle through A is also the locus of a point from which two circles, with centres B, C and radii c,b respectively, are 'seen at a constant angle'.

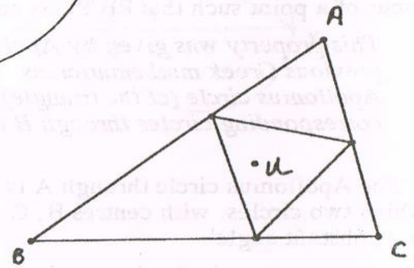
This property is the theme of one of the animated geometry films of Jean-Louis Nicolet. The angle between the two tangents to a circle from a point may be taken as a measure of the apparent size of the circle when viewed from the point. It is interesting to note that from our point of view the moon and the sun appear to be about the same size - they are both 'seen at a constant angle' of about .5° (see note with proof 15.3.)

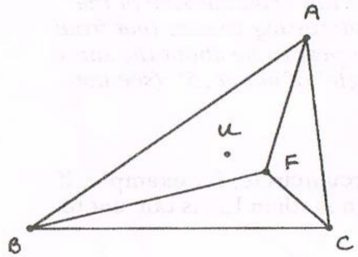
• An Apollonius circle is orthogonal to the circumcircle; for example, if L is the centre of the Apollonius circle through A, then LA is tangent to the circumcircle.

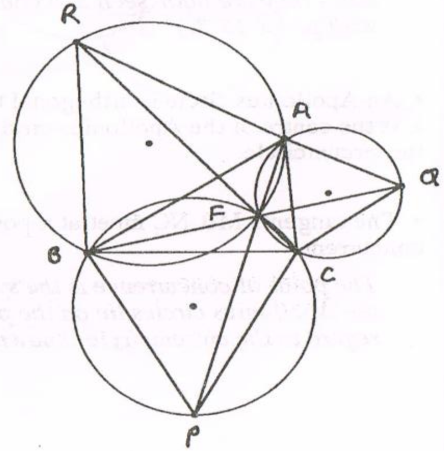
• The tangents MB,NC meet at a point K_1: AK_1, AK_2,AK_3 are concurrent.

The point of concurrence is the sytmmedian point K; the centres of the Apollonius circles lie on the polar of the symmedian point with regard to the circumcircle, known as the Lemoine line (cf 6.2).









15: Fermat points

- The three Apollonius circles of a triangle intersect at two points U,U'. The centres of the circles lie on the mediator of UU'.

  The common points of the Apollonius circles are sometimes called the isodynamic (or Hesse) points of the triangle.

- The pedal triangles of the isodynamic points are equilateral.

- The isogonal conjugates of U,U' are points at which the sides of the triangle ABC subtend angles of 60° or 120°.

  The isogonal conjugates of the isodynamic points are known as the Fermat points F,F' of the triangle (or sometimes as the isogonal centres). When the angles of the triangle do not exceed 120° the internal Fermat point is the point with minimal sum of distances from the three vertices.

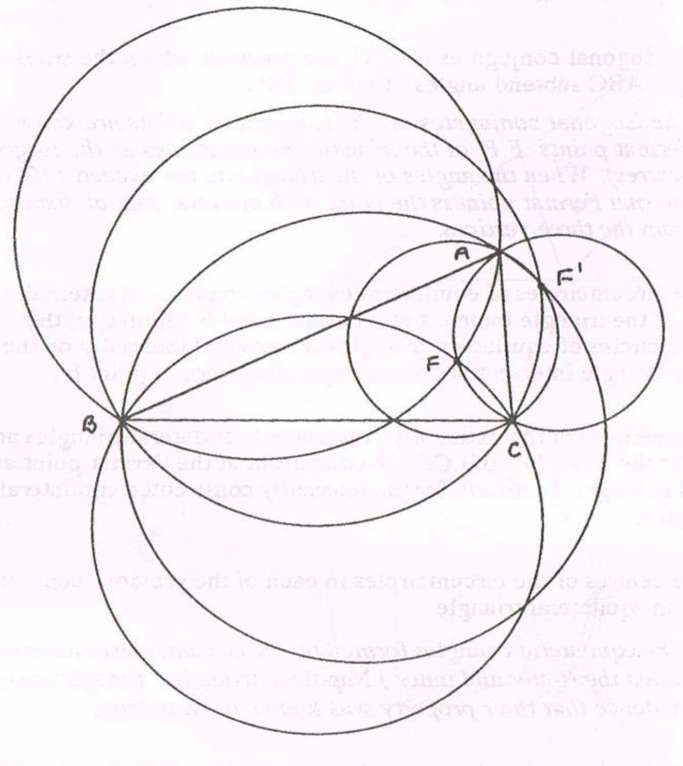
- The circumcircles of equilateral triangles constructed externally on the sides of the triangle intersect at a Fermat point F. Similarly, the circumcircles of equilateral triangles constructed internally on the sides of the triangle intersect at the corresponding Fermat point F'.

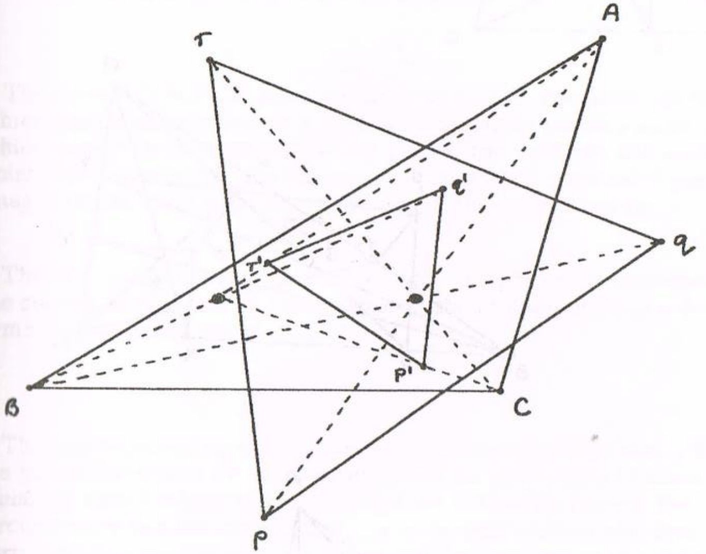
- The vertices of the externally constructed equilateral triangles are P,Q,R: the lines AP, BQ,CR are concurrent at the Fermat point and are equal in length. Similarly for the internally constrcuted equilateral triangles.

- The centres of the circumcircles in each of the previous constructions form an equilateral triangle.

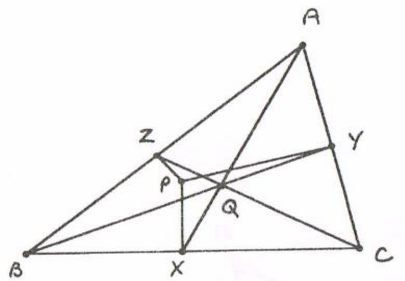
  The equilateral triangles formed by the circumcentres have been called the (outer and inner) Napoleon triangles, though there is no evidence that their property was known to Napoleon.

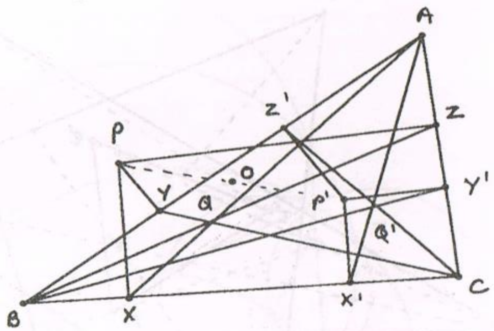
- The Napoleon triangles are each in perspective with the triangle ABC.

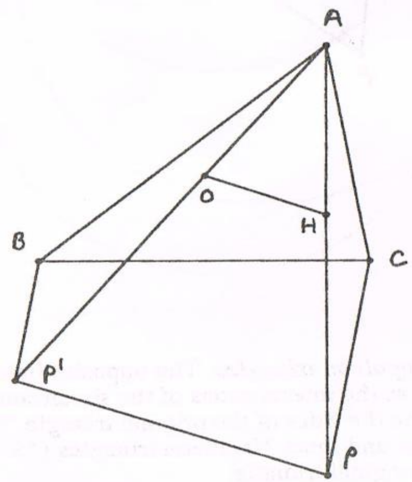




Fermat points and Napoleon triangles: The opposite figure shows the Fermat points (15.4) as the intersections of the six circumcircles of the equilateral triangles on the sides of the original triangle. The above figure shows the outer and inner Napoleon triangles (15.7) each in perspective with the original triangle.







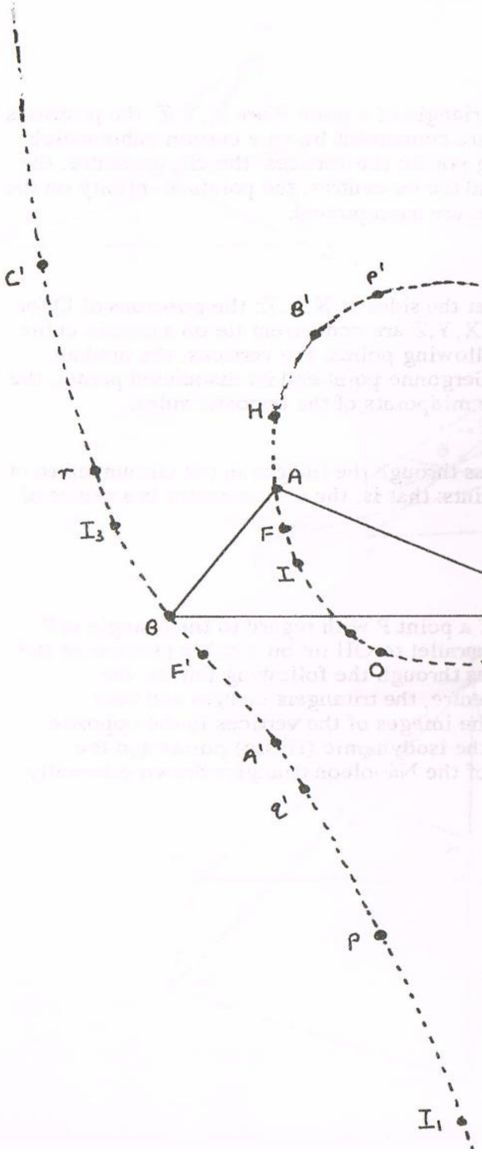
16: Some cubics

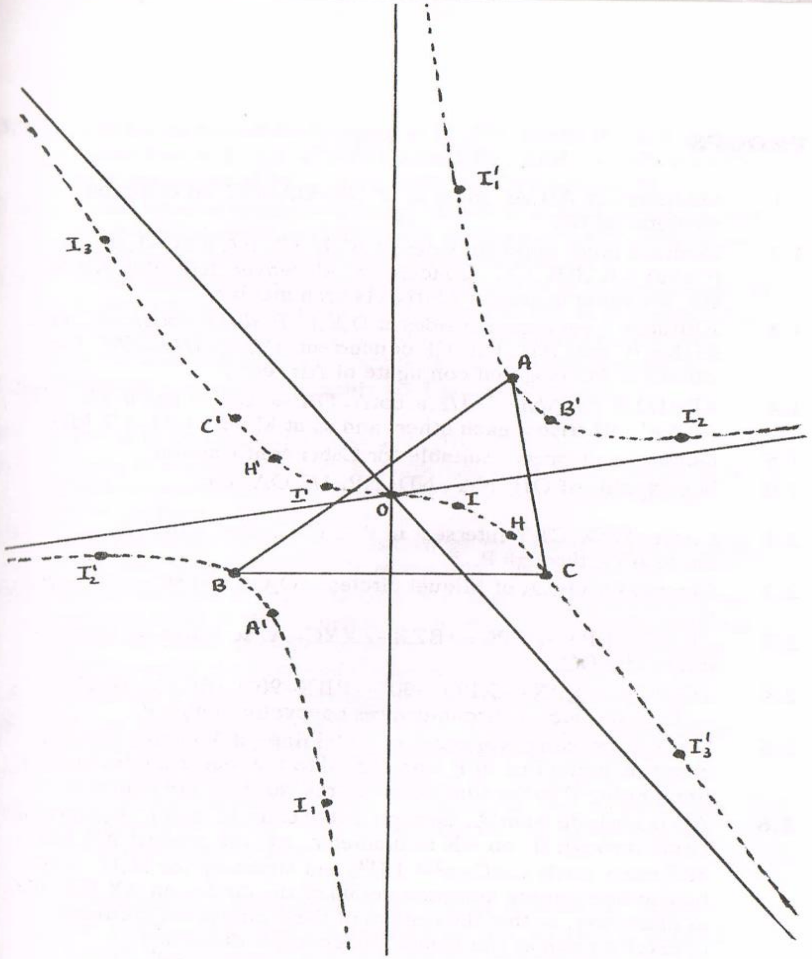
• The vertices of the pedal triangle of a point P are X,Y,Z: the positions of P for which AX,BY,CZ are concurrent lie on a certain cubic which passes through the following points: the vertices, the circumcentre, the orthocentre, the in-centre and the ex-centres, the points at infinity on the mediators (so that these lines are asymptotes).

• The lines AQ,BQ,CR meet the sides at X,Y,Z: the positions of Q for which the perpendiculars at X,Y,Z are concurrent lie on a certain cubic which passes through the following points: the vertices, the median point, the orthocentre, the Gergonne point and its associated points, the images of the vertices in the midpoints of the opposite sides.

• The above cubics each pass through the images in the circumcentre of the corresponding sets of points: that is, the circumcentre is a centre of symmetry for each curve.

• The isogonal conjugate of a point P with regard to the triangle is P': the points for which PP' is parallel to OH lie on a cubic (*known as the Neuberg cubic*) which passes through the following points: the circumcentre and the orthocentre, the tritangent centres and their corresponding conjugates, the images of the vertices in the opposite sides and their conjugates, the isodynamic (Hesse) points and the Fermat points, the vertices of the Napoleon triangles drawn externally and their conjugates .





Cubics associated with the triangle (16.1/4): The above figure shows the locus of a point whose pedal triangle is in perspective with the original triangle. The opposite figure shows the locus (Neuberg's cubic) of a point whose join with its isogonal conjugate is parallel to the Euler line of the triangle.

PROOFS

1.1 Mediators of AB, AC meet at O: OB=OA=OC, so O lies on mediator of BC.

1.2 Medians meet opposite sides at A', B', C': BA'/CA'=1, etc, so (Ceva) AA', BB', CC', concurrent. Moreover, triangles AB'M, etc, are equal in area $\Rightarrow$ M trisects each median.

1.3 Altitudes meet opposite sides at D, E, F: BD/CD=cotB/cotC, etc, so that (Ceva) AD, BE, CF concurrent. (Note $\angle$DAC=90°−C=$\angle$BAO $\Rightarrow$ AO isogonal conjugate of AH, etc.)

1.4 AP=1/2.c.cosA/sinC=1/2.a.cotA=OA'$\Rightarrow$ OA' = and $\parallel$ 1/2.AH $\Rightarrow$ AA', OH trisect each other, and so at M with OM=1/2.MH.

1.5 *Gossard's theorem* - suitable for Cabri confirmation.

1.6 N midpoint of OH: NA'=ND=NP=1/2.OA, etc.

2.1 Circles BZX, CXY intersect in P: $\angle$YMZ=B+C=180°-A $\Rightarrow$ circle AYZ through P.

2.2 Centres O_1, O_2, O_3 of Miquel circles: $\angle O_2O_1O_3$=180°−$\angle$ZPY=A, etc.

2.3 $\angle$BPC=$\angle$BPX−$\angle$XPC=$\angle$BZX−$\angle$XYC=A, so P lies on circum-circle of ABC.

2.4 $\angle O_2PO_3=\angle O_2PX+\angle XPO_3$=90°−$\angle$PBX−90°+$\angle$PCX=$\angle$BPC=A =$\angle O_2O_1O_3$, etc, $\Rightarrow$ circumcentres concyclic though P.

2.5 See 2.6, or using properties of pedal line (cf 3.1) note that the common pedal line of P with regard to the four triangles bisects line joining P to the four orthocentres, so these are collinear.

2.6 AD is altitude from A, through orthocentre H, and L is centre of circle through H, on AX as diameter, etc: the product AH.HD = $4R^2$.cosA.cosB.cosC=a^2/4-LH^2, and similarly for M,N: $\Rightarrow$ H lies on line joining common points of the circles on AX, BY, CZ as diameters, so that the centres of these circles are collinear. (*This is known as the Gauss-Bodenmiller theorem.*)

2.7 $\angle$PYA=$\angle$PZA=90° $\Rightarrow$ PA is diameter of circle AYZ, etc.

2.8 Successive pedal triangles are $X_iY_iZ_i$: $\angle B_3A_3P=\angle B_2C_2P$ = $\angle A_2B_1P$ =$\angle A_2B_1P$=$\angle$BAP, similarly $\angle PA_3C_3$=$\angle$PAC, so that $\angle B_3A_3C_3$=$\angle$BAC.

3.1 $\angle$BZX=$\angle$BPX=90°−$\angle$PBC=90°−$\angle$PAY=$\angle$APY, so XYZ a line.

3.2 AH meets circumcircle again at H', PH' meets BC at S and pedal line at T: $\angle$YXP=YCP=$\angle$AH'P=$\angle$XPH' $\Rightarrow$ TP=TX=TS, ie T midpoint of PS $\Rightarrow$ XY$\parallel$ ZHS, so XYZ bisects PH.

3.3 Pedal lines of diametrically opposite P, P' are perpendicular and meet medial circle at Q, Q': Q, Q' are diametrically opposite so pedal lines through Q, Q' meet on the medial circle.

3.4 Pedal lines of PQ meet at X, H' image of H in X, orthocentre of PQH' is R: pedal lines of P, Q bisects PH, QH and so parallel to PH', QH', $\Rightarrow$ $\angle$PRQ=180°-$\angle$PAQ, $\Rightarrow$ R on circumcircle and pedal line of R$\parallel$ RH' and hence the third pedal through X.

3.5 *Steiner's theorem* - suitable for Cabri confirmation.

4.1 For A is on altitudes of triangle HBC, etc.

4.2 O_2O_3 is mediator of AH, so perpendicular to OO_1, etc $\Rightarrow$ O is orthocentre of $O_1O_2O_3$; moreover OO_1=AH so that circum-centres form a set congruent to A, B, C, H.

4.3 Medial circle of BHC passes through the midpoints of BC, CH, BH, and so is medial circle of triangle ABC.

4.4 Equal weights at the vertices A, B, C and orthocentre H may be replaced two at the midpoint A' of BC and two at the midpoint P of AH; and the centre of gravity of these at the midpoint of A'P, namely the centre of the medial circle.

4.5 *Beltrami's theorem*: an application of 4.4.

4.6 Equal weights at the vertices A, B, C, D may be paired in three ways each of which yields the common centre of gavity.

4.7 Concyclic points A_i ; orthocentre of A_2, A_3, A_4 is H_1, etc: $A_1H_4H_1A_4$ is a parallelogram, $H_1H_4=A_1A_4$, etc, ie the H_i are congruent to the A_i , and $A_1A_2A_3H_4$, etc, or $A_1H_2H_3H_4$, etc, form orthocentric sets.

5.1 Distances of X from AC, AB are y, z, of A' from AC, AB are q, r: y/AX=r/AA' and z/AX=q/AA' $\Rightarrow$ y/z=r/q=b/c $\Rightarrow$ BX/CX =c^2/b^2, etc, so that (Ceva) AX, BY, CZ concurrent.

5.2 Reflecting in an angle bisector, the corresponding symmedian and antiparallel become median and parallel.

5.3 Distances of K from sides are x, y, z: y/z=b/c, etc, (cf 5.1) $\Rightarrow$ x/a=y/b=z/c=2abc/R($a^2+b^2+c^2$).

5.4 $\angle MAB=\angle KAY=\angle KZY=90°-\angle AZY$, so that AM is perpendicular to YZ; with A'' the image of A in A', triangles CAA'', KYZ are similar, and since these are rotated through 90° and CA' bisects AA'', then XK bisects YZ.

5.5 The centres of rectangles inscribed in the triangle with a side on BC are collinear; and the midpoint of BC, the symmedian point K and the midpoint of the altitude through A are such centres. (*Brocard's theorem*)

5.6 Alias, the median point lies on the Euler line of the pedal triangle of the symmedian point; for the pedal triangles of the isogonal conjugates M and K have the same circumcircle, with its centre at the midpoint of MK. (*Tucker's theorem*)

5.7 Alias, the symmedian point, the incentre, and the symmedian point of the ex-centres. are collinear. (*Van Aubel's theorem*) The isotomic conjugate of the orthocentre also lies on the line.

6.1 $A'K_1, K_2K_3$ meet BC at X,X': $BX'/\sin C=AX'/\sin B$ and $CX'/\sin B=AX'/\sin C \Rightarrow BX'/CX'=c^2/b^2=BX/CX \Rightarrow AK_1$ passes through symmedian point K.

6.2 The polar of X' is AX through K $\Rightarrow$ polar of K through X'.

6.3 AY_3KZ_2 is a parallelogram, so AK bisects Y_3Z_2; hence Y_3Z_2 is antiparallel to BC, $\angle AY_3Z_2=\angle AZ_1Y_1$, etc, ie the six points are concyclic, with centre at midpoint of OK.

6.4 See previous note.

6.5 BV,CW meet at T, distant x from BC, etc: so $x:y=X_3X_2:Y_1Y_3$ and $x:z=X_3X_2:Z_2Z_1 \Rightarrow y:z=Y_1Y_2:Z_1Z_2$, so that T lies on AU.

6.6 For U'V'W' is Pascal line of the six points on the circle.

6.7 For KA bisects Y_3Z_2, etc.

7.1 $\text{Cot } w=1/2.a/KK_1=(a^2+b^2+c^2)/4D=\cot A+\cot B+\cot C$.

7.2 $\cot^2 w-3=1/2.\Sigma(\cot B-\cot C)^2 \Rightarrow \cot w\geq\sqrt{3} \Rightarrow w\leq 30°$.

7.3 $BX/\sin w=AX/\sin B$ and $CX/\sin(A-w)=AX/\sin C \Rightarrow BX/CX =c^2/a^2$, etc, so that (Ceva) AR,BP,CQ concurrent.

7.4 $\angle WBC=\angle BAW=\angle WCA \Rightarrow$ AC tangent to circle BWC, etc.

7.5 AW,BK,CM meet sides at X,Q,C': $BX/CX=c^2/a^2$ as above; $CQ/AQ=a^2/c^2$; $AC'/BC'=1$; so (Ceva) AX, BQ,CC' concurrent.

8.1 $BX/\sin\theta=c/\sin(B+\theta)$, $CX/\sin(A-\theta)=b/\sin(B+\theta)$, $BA_1/\sin\theta =$ $AA_1/\sin(B-w)$, $CA_1/\sin(A-\theta)=AA_1/\sin(C-w)$, $\Rightarrow BX/CX= c.\sin(B-w)/ b.\sin(C-w)$, etc; so (Ceva) AX,BY,CZ concurrent.

8.2 Triangles ABC and $A_1B_1C_1$ are similar (cf 8.2) and in perspective, so have a common median point.

8.3 $\angle B_1OC_1=A=\angle B_1A_1C_1$, so O on circle $A_1B_1C_1$; and the circle on OK as diameter passes through A_1,B_1,C_1.

8.4 OK is diameter of Brocard circle; midpoint of OK is also centre of Lemoine circle, as above (5.6)

8.5 $\angle BA_2C=2A \Rightarrow A_2$ on circle BCO $\Rightarrow$ perpendiculars from A_2 to AB,AC are in ratio c:b, so that A_2 lies on symmedian AK; and $\angle OA_2A=90° \Rightarrow \angle OA_2K=90°$.

8.6 As proved in previous note.

8.7 $\angle C_1A_1M=\angle CAM=\angle BAK=\angle CAA_2=\angle C_1A_1A_2 \Rightarrow$ points A_1, M, A_2 are collinear.

9.1 P,Q,R are on circumcircle: $BX/\sin(C-A)=BR/\sin(B-C))$, $CX/\sin(A-B)=CQ/\sin(B-C)$, $\Rightarrow BX/CX=b.\sin(C-A)/c.\sin(A-B)$, so (Ceva) AP,BQ,CR concurrent.

9.2 This and the following properties are suitable for Cabri confirmation. Otherwise they may be verified using Argand representations of the points; for example, the Steiner point has affix s=abck/k where k is that of the symmedian point, whence the various properties may be derived.

9.3 The Argand representations of the vertices of the Brocard triangles are $a_1= k/2+bck/2$, etc.

9.4 The other two vertices are s_2, s_3 with $s+s_2+s_3=a+b+c$.

9.5 Steiner sought the circumellipse with minimum area, and found that this was the one with centre at the median point.

10.1 This and the following properties may be derived from the corresponding properties of the Steiner point.

10.2 In the Argand representation the affix of the Tarry point is t=-s.

10.3 This is also clearly suitable for Cabri confirmation.

10.4 Affix of the Euler point is abch/h where h is that of the orthocentre.

10.5 Affix of the derived point is abc(3k-a-b-c)/3(k-s).

11.1 Bisectors of B,C meet at I: I equidistant from sides AB,AC, and so on bisector at A.

11.2 External bisectors at B,C meet at I_1: I_1 equidistant from sides AB,AC, and so on bisector at A.

11.3 Trigonometric manipulations yield $IN^2=(R/2-r)^2$, so that medial circle, centre N, radius R/2, touches circle centre I, radius r, etc. *(Feuerbach's theorem: there are many other proofs.)*

11.4 I_1A is perpendicular to I_2I_3.

11.5 This is an application of Feurbach's theorem (11.3).

11.6 The distances from circumcentre to sides are α,β,γ,δ, and the circumradius is R: then the sum of the radii of the in-circles touching a diagonal is α+β+γ+δ-2R, ie the same in each case. It can then be shown that the in-centres form a rectangle.

12.1 AY=AZ=s-a, etc, so (Ceva) AX,BY,CZ are concurrent.

12.2 $BX_1=BZ_1=s-c$, $CX_1=CY_1=s-b$, $AY_1=AZ_1=s$, so (Ceva) AX_1,BY_1,CZ_1 are concurrent.

12.3 A,B,C are exsymmedian points (cf 6.1) of the triangle XYZ, so that g is the symmedian point of XYZ, etc.

12.4 $BX_1/CX_1=(s-c)/(s-b)$, etc, so (Ceva) AX_1,BY_1,CZ_1 are concurrent.

12.5 BX/CX=(s-b)/(s-c), $CY_3/AY_3=s/(s-b)$, $AZ_2/BZ_2=(s-c)/s$, so (Ceva) AX,BY_3,CZ_2 are concurrent.

12.6 $BX_1=CX=(s-c)$, etc.

12.7 $X_1D/A'X=AD/IX=2s/a \Rightarrow AX_1/IA'=2s/a$; but $AX_1=s/a$. An so that An = and ∥ 1/2.IA', ⇒ IA', AA' trisect each other, and so at the median point M.

13.1 BX1=2C'X'=(s-c), ⇒ AX1 through Nagel point n.

13.2 S midpoint of In ⇒ M midpoint of SI ⇒ A'S ∥ AI ⇒ A'S bisector of angle B'A'C'.

13.3 Each side may be replaced by a weight equal to half the side at the midpoint, ie equal to the opposite side of the midpoint triangle.

13.4 Triangles IQ'R', ABC are similar, scale factor 1/2.

14.1 XB/XC=AB/AY=AB/AC.

14.2 ∠XAX'=90°, so that XX' is diameter of circle through A; for any other vertex P than A, PB/PC=XB/XC=X'B/X'C=constant ⇒ X,X' are fixed points, so that P lies on the same circle as A.

14.3 PB=c.sinβ/2, similarly PC=b.sinγ/2: then β=γ ⇒ PB/PC=c/b =constant. *Note: Moon: distance=384,400km, radius=1738km; Sun: distance=149,598,000km, radius=696,000km; so that subtended angle for Moon is .52°, and for Sun is .53°.*

14.4 ∠LAC=∠LAX-A/2=∠LXA-A/2=∠ABL.

14.5 Polar of K_1 through L, so polar of L through K_1 ie AK_1 is polar of L, hence through pole of collinear LMN.

15.1 U on A-circle through B,C: UC/UA=BC/BA, UA/UB= CA/CB ⇒ UB/UC=AB/AC, so that U also on A-circle through A.

15.2 Sides of pedal triangle are UA.sinA,UB.sinB,UC.sinC; these are equal since UA:UB:UC=1/a=1/b=1/c.

15.3 ∠FBC+∠FCA= ∠UBA+∠UCA=60° ⇒ ∠BFC=120°.

15.4 Circumcircles ABR,ACQ meet at F: ∠BFC=120° ⇒ F on circle BCP; and similarly.

15.5 Lines FA,FP,etc, all at 60°, so that A,F,P collinear, etc. Also, $AP^2=a^2+c^2-2ac.\cos(B+60°)=(a^2+b^2+c^2)/2+8/3abc$, which is symmetrical in a,b,c so that AP is equal to BQ,CR.

15.6 Circumcentres p,q,r: lines pq,pr perpendicular to CF,BF ⇒ ∠qpr =180°-∠BFC=60° ⇒ triangle pqr equilateral.

15.7 Ap/sin(B+30°)=c/sin(θ+30°), Ap/sin(C+30°)=b/sin(θ-30°), BX/sin(θ+30°)=pX/sin30°, CX/sin(θ-30°)=pX/sin30°, BX/CX =c.sin(B+30°)/b.sin(C+30°), so (Ceva) Ap,Bq,Cr concurrent.

16.1 The particular points may be immediately verified from the definition; for example, the vertices of the pedal triangle of the circumcentre are A',B',C', and AA',BB',CC' are concurrent.

16.2 For example, the medians AM,BM,CN meet the sides at A',B', C' and the perpendiculars to sides at these points are concurrent.

16.3 If P is on the first curve, then AX,BY,CZ are concurrent at Q; the isotomic conjugate of Q, namely Q', will be such that AQ', BQ', CQ' meet sides at X',Y',Z' and the perpendiculars to sides at these points meet at a point P'; since BX'=CX, etc, P'X' is image of PX in mediator of BC, ie P' is image of P in O.

16.4 For example, if P is the image of A in BC then its isogonal conjugate P' is such that P'BA=180°-B; so AP'/sinB=c/cosA ⇒ AP'/AO=sinBsinC/cosA=AP/AH ⇒ PP' parallel to OH.