**Students' conceptualisations of function revealed through definitions and examples**

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*This study aims to learn about the conceptualisations of function that students express when they are responding to fictitious students’ statements about functions. We also asked them what is meant by 'function' and many voluntarily used examples in their responses. The task was developed in collaboration with teachers from two curriculum systems, England and Israel. It was given to 10 high-achieving English students from each of the years 10–13 and to 10 high-achieving Israeli students from comparable years* *(total of 80 students). Data analysis included identifying students' dominant ideas for functions as expressed in their responses, and analysing the types of examples that students used to explain their responses. Differences found between countries led to conjectures about the influence of curriculum and teaching, and in particular, about the role of word, in this case 'function', in concept image development.*

Key words: functions; conceptualisations of function; example spaces; curriculum

**Introduction**

Function is one of the key ideas in mathematics, the diversity of its interpretations and representations spreading across pure and applied mathematics. Students’ mental concept images of functions may be different from mathematical definitions (Vinner & Dreyfus, 1989). One relevant question is to what extent the concept images are curriculum-dependent. Concept images are influenced by the examples to which students have been exposed in school (Vinner, 1983) and prototype examples (Schwarz & Hershkowitz, 1999). The learning experiences of students according to teachers’ decisions and choices of contexts and resources, and the structure of the curriculum, are of interest to us in this qualitative exploratory study. We posed the same questions in two countries, England and Israel, to a small sample of students so we could study responses in depth and make conjectures about the complexity of their understanding of functions.

We aim -

* to learn about school students' conceptualisations of function when:
  + - responses are prompted to fictitious students’ statements about functions;
    - they are asked what, for them, is meant by 'function';
    - they voluntarily use examples in their responses.
* to conjecture the possible connections with curriculum and teachers' expectations.

**Theoretical background**

The concept of function is of fundamental importance in the learning of mathematics and has been a major focus of attention for the mathematics education research community for several decades (e.g., Dubinsky & Harel, 1992; Sfard, 1991, 1992; Sierpinska, 1992; Slavit, 1997; Tall, 1992; Vinner & Dreyfus, 1989). Various approaches have been offered to explore the concept of function in mathematics teaching and learning. There is the historical notion of a function, developed in the 17th century, as a rule-based relationship between two interconnected numerical variables. The emphasis was on how changes in one variable (usually called the dependent variable) correlated with changes in another variable (usually called the independent variable). There is also the more recent ordered pair definition of function, which is often referred to as the Bourbaki approach. This formal set-theoretic definition is very different; in this approach, a function idea is not only associated with numbers, and the notion of dependence between two variables is implicit (Markovits, Eylon, & Bruckheimer, 1986).

We see these historical perspectives as related to two approaches to understanding functions: correspondence and covariation (e.g., Confrey & Smith, 1994, 1995; Leinhardt, Zaslavsky, & Stein, 1990). The correspondence approach associates a unique *y*-value with any given *x*-value, thus building a correspondence between *x* and *y*. The correspondence approach is emphasised in teaching methods such as mappings and input-output models. It can facilitate the introduction of the concepts of domain and range, as well as the notions of onto and one-to-one (Selden & Selden, 1992), and makes a plausible contribution towards understanding that relations between two sets of numbers might (sometimes) be expressed as general algebraic ‘rules’ by which an input value gets converted into a particular output value. The correspondence approach and the ordered pair definition are technically similar; however the former seems to be easier for students to understand than the latter, which can be seen as an encapsulated expression of the former. Both of these perspectives could be said to focus on a point-wise perception of functions as the emphasis for learners at first is on the relation between particular instances or subsets of the variables, either as numbers, objects or expressions. A covariation approach to functions involves understanding the manner in which a change in one variable is related to a change in another, or how those variables change together. In its strongest form, this involves seeing each variable as a parametric function rather than values in one set being dependent on other sets (Confrey & Smith, 1995; Saldanha & Thompson, 1998). In a more common form, this is about comparing changes in dependent variables to changes in independent variables as a rate of change or gradient (Carlson & Oehrtman, 2005). This form relates to modelling phenomena where there is physical dependence, and also to the mathematical convention that the horizontal axis represents *x*, and this is the independent variable from which *y*-values are generated so that a move in the x-direction is associated with a change in the y-ordinate. Analyzing, manipulating, and comprehending the relationships between changing quantities illustrates the covariation view, which could engender an understanding of functions as the relations between two sets of varying values, rather than actions on individual values, and possibly as graphs illustrating instantaneous rates of change of variables. A dual view of functions as representing both correspondence and covariation allows the student to better understand actions performed on a function, such as a ‘shift’ translation (e.g., changing *f*(*x*) to *f*(*x* + 3)) or taking a derivative (Slavit, 1997).

Functions can be viewed in terms of action or process conceptions (Carlson & Oehrtman, 2005; Dubinsky & Harel, 1992). An action view relates to the computational aspects associated with functions, such as an arithmetic process or an input-output 'function machine'. A process view involves "a dynamic transformation of objects according to some repeatable means that, given the same original object, will always produce the same transformed object" (Breidenbach, Dubinksy, Hawks and Nichols, 1992, p.251). For example, one can consider the function *f*(*x*) = 4*x* + 6 to be a series of instructions used to compute numeric values for a given input, or as a process that transforms any input value from the domain in a specific way, but in either case it is the process that is the focus and not the inputs, outputs and operations. Breidenbach et al. (1992) claim that, consonant with APOS theory, students are more able to understand properties such as one–one, onto, and inversion once a process conception is achieved as it provides an entryway into an object-oriented understanding of function. This shows up through the ability to talk about and treat functions as reified mathematical objects in their own right and thus being able to classify, combine and transform them in various ways (Sfard, 1991). Another helpful classification is that functions can be approached either point-wise (as a collection of points) or globally. Reading values from a given graph, generating coordinates and point-plotting are examples of point-wise approaches to functions. The point-wise approach could be considered over-simplistic for students, in contrast to thinking of the behavior of a function in a global way as a process (Even, 1990), and also in contrast to thinking covariationally.

Research suggests that school-level shifts between views of functions might be hierarchical. Tall (1992), for example, referred to shifts between seeing functions as instructions to calculate one numerical set from another, to seeing them as correspondences between sets, to seeing them as describing variations, to using functions to describe situations, and also to understanding functions (often in graphical form) as objects in their own right. Ronda (2009) identified a sequence of ‘growth points’ in students’ understanding of linear functions which can apply to functions more generally: Equations might first be experienced as formulae or procedures for generating values; then as representations of relationships; then representing properties of relationships; and finally functions are mathematical objects that can be manipulated or transformed.

Vinner and Dreyfus (1989) show that students’ mental images of functions may be different from mathematical definitions. They are frequently based on a concept image which refers to "the set of all the mental pictures associated in the student’s mind with the concept name, together with all the properties characterizing them" (Vinner & Dreyfus, 1989, p. 356). Several studies have shown that in many cases the function concept images that are brought into use by students are limited. Dreyfus and Eisenberg (1983) reported on college-level students' general idea that functions are continuous, smooth and calculable. Several studies have also suggested that the learning experiences have an effect on students' conceptions. Markovits et al. (1986) suggested that discontinuous functions, functions with split domains, and the constant function, need to be introduced to extend students' previous experience and lead to a need for a formal definition. But students have difficulty abandoning action and pointwise views of functions if that is how they first met them (Breidenbach et al., 1992; Meel, 1998). Seeing functions only as mappings between sets of discrete data can reinforce a view that a function is a set of discrete data points. It can also reinforce a view that anything that can be mapped is a function (Spyrou & Zagorianakos, 2010). College-level students in these studies are generally using the word 'function' and a related personal concept image; in our study of school children, not all of them have the word 'function' attached to their pre-function-concept experiences, as we shall show.

We are most interested in the effects of the introduction of the word ‘function’ at different stages in students’ mathematics: in grade 7 in Israeli schools and in year 12 in English schools. We will return to the issue later, but at this point we wish to emphasise that the essential role of theword in concept development, as sign, has been conjectured and studied since Vygotsky (1986; see also Wertsch, 2007). The association of concept images with the mediated role of theword early in the learning of functions may well facilitate richer connections and meaning during students’ development than an apparently fragmented set of activities building concept images that are associated by a unifying term later in the curriculum. As the notion of concept image is much better known in the field than the discussion around word and sign systems we will continue to use concept image and will reserve further discussion for later in the paper.

From the preceding discussion, the dimensions of the students' concept image include: action/process; point-wise/global; correspondence/covariation, with process, global and covariation being more closely associated with eventually understanding functions as objects. One relevant question is to what extent these shifts in meaning and the concept images are curriculum-dependent. Concept images are influenced by the examples to which students have been exposed in school (Vinner, 1983). Curricula are designed with a particular conceptual progression in mind, so it is not surprising if students display similar order of learning as their curriculum. The example space students can draw on for functions is limited by their experience of functions, and tends to be orientated around prototype examples as used by teachers and textbooks (Schwarz & Hershkowitz, 1999).

These two aspects of the learning process, the learning experiences of students according to teachers’ decisions and choices of contexts and resources, and the structure of the curriculum, are of interest to us in this study. We will return to discuss our view in the concluding section, following the analysis of the data, but we wish to point out at this stage that we take a Vygotskian perspective on learning whereby what students learn is internalization from what precedes them on the social plane. Language is critical here and will therefore play an important role in our reflections, and it is also of importance that the 'social' includes the teacher's choice of examples of functions, and how their features are emphasised or alternatively taken-for-granted, since this is how students will also organise their knowledge, responses and discourse. This connection is made by Vinner (1983) and we would go further to say it also explains that, even if a student's concept image is well-populated, they may still be socialised into showing limited 'normal' knowledge in communication. Coming from a more Piagetian perspective, Dubinsky also recognised that response to a situation is not the same thing as revealing one's total potential repertoire (Dubinsky, 1989, p. 2):

An individual’s mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context and constructing or reconstructing mathematical actions, processes and objects and organising these in schemas to use in dealing with the situations.

We did not ask students to provide examples, nevertheless the extent of example use in students' responses led us to analyse these in terms of personal example spaces (PES) (Watson & Mason, 2005; Sinclair, Watson, Zazkis, & Mason, 2011). PES usually do not conform to conventional example spaces but instead are constructed through a learner's experience. They are structured in terms of their contents, generality, and inner connections (Sinclair et al., 2011) and hence their use is situated, being triggered by features of a task (Watson & Mason, 2005). It is generally observed that learners find it hard to create counter-examples to statements, often seeing non-examples as exceptions rather than as refutations (e.g. Zaslavsky & Ron, 1998). Sometimes the difficulty appears to be in the logic of refuting a rule, but it can also be the result of too limited a PES (Zaslavsky & Peled, 1996; Watson & Mason, 2005).

This study, therefore, aims to provide detail of these effects from two countries, England and Israel, whose curriculum expectations about functions concepts are very different. We posed the same questions in each country, and prepared these with input from teachers from both countries in order to ensure cultural validity and curriculum fidelity. The study aims to learn about school students' conceptualisations of function through: responses to fictitious students’ statements about functions; being asked directly what, for them, is meant by 'function'; and their voluntarily use of examples in their responses to these two tasks. We then conjecture the possible connections with curriculum and teachers' expectations.

**The task**

The task focuses on students’ ways of seeing functions and was intentionally designed to stimulate students' thinking of various ideas and concepts related to functions, by offering definitions and quasi-definitions of several kinds.

The task includes two parts. The first part presents a situation of a group of five fictional students discussing what a function is, each one of them posing her or his idea:

* Arthur said: I see functions as input-output machines, which receive some input and give an appropriate output.
* Ruth said: I see function as a mapping of each element of one set to exactly one element of a second set.
* Ian said: Functions for me represent relations between variables.
* Naomi said: A function shows how one variable changes in relation to another variable.
* Liz said: I see functions as expressions to calculate *y*-values from given *x*-values.For example, *y* = 4*x* + 7.

Students were asked to read each of the students’ ideas, and to write their response to it in the following way (illustrated with respect to Arthur's statement):

Which one of the following statements reflects your thinking about Arthur’s description of functions? Mark your response and explain your choice.

|  |  |
| --- | --- |
|  | All functions fit Arthur’s description. |
|  | Some functions fit Arthur’s description. |
|  | Arthur is wrong. |

Explanation: **\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

The second part of the task asked for the student's own idea for function:

* Now, after you have responded to the students' ideas, write what is a function for you.

The specific design of the task aimed at prompting students' thinking about each of the various ways of seeing functions, hoping they would "dig" into the world of functions that is available to them to support or contradict these views of functions, and look for generalities and restrictions. We chose these five ideas of functions to reflect the variety of conceptualisations they might have met and constructed, according to the teachers, curriculum and research (Watson, 2013). Arthur's idea of input-output emphasises point-wise and action conceptions; Ruth's emphasises sets, relations and mapping as a process; Ian's emphasises relations between variables, patterns and dependency; Naomi's idea we hoped would draw out any ideas about covariation; Liz's would draw on students' experience of algebraic functions. Of course these ideas are not mathematically distinct, and the use of fictional characters to present them was intended to engage students in thinking about each description as they would if someone had said it in class. The fact that most respondents were positive about more than one of these statements showed that they had not assumed that only one could be chosen.

The second part of the task aimed at examining what students chose to write, after considering the various ideas, as their own idea of function, which of course may have been challenged in the previous questions.

The task script was developed with the help of native speaking teachers in English and Hebrew so that the language expectations were similar for all students. The author team contains bi-linguists with strong mathematical credentials so we know that the concepts were faithfully reproduced in each language and were expressed in ways that would be expected in each language. There are no mathematical conceptual differences arising from cultural background. Answers from Hebrew students have been translated into English by the research team and there were only two instances where translations caused further discussion of underlying meanings.

We acknowledge that the way the task was designed may prompt some ideas of functions while ignoring others, but this is inevitable in maintaining curriculum fidelity.

**Israeli and English curricula**

A direct comparison of the English and Israel curriculum for functions year by year is not possible, because the English curriculum at the time of the study was described in terms of levels[[1]](#footnote-1), where the Israeli curriculum was in terms of years. However, a comparison of content order is possible. In addition, non-statutory year-by-year guidance was given to teachers in England and we have taken that into account.

As a country with a centralised educational system, the Israeli school curriculum is developed and regulated by the Ministry of Education, and textbooks have to be officially approved. The national mathematics curriculum in Israel for secondary schools (Ministry of Education, 2009) provides detailed recommendations related to teaching functions. The curriculum on functions starts in year 7 (age 12) with explicit use of the word 'function' in the context of numerical functions described verbally, graphically, through tables of values and algebraic expressions, and is defined in year 7 as the matching of a unique number to each number we choose. These matchings can be represented graphically, but students also meet domains that exclude some numerical values. The idea of 'mapping' to connect ordered pairs is not used. The rate of change is the quotient of changes in y and changes in x values; there is discussion of constant and non-constant rates of increase and decrease and their graphical representations. Constant functions are included in this discussion. Students discuss graphs and rates of change explicitly in the context of realistic phenomena. The function notation is introduced for 12 to 14 year olds, either before or during a formal treatment of linear functions.

The typical Israeli student would therefore know that there are things called 'functions' to be studied, and they are to do with matching numerical variables according to some rule and can be represented on graphs, and sometimes algebraically, and can be represented symbolically. The rate of change of one variable in relation to the other is calculated using quotients of change on an interval, and can be constant, zero or changing; change can be increasing or decreasing. Linear functions would be presented as a special class of functions, having special features.

Older students in both countries then have a formal treatment of quadratics, combinations and transformations of functions, and calculus. The English curriculum and testing system does not expect the word 'function' and its notation to be used until years 11/12 for higher attaining students when they begin to transform them, and hence need to treat them as objects. Earlier introduction to functions is through generalising linear and quadratic sequences and studying mappings that connect domain values to range values. These mappings then give rise to data tables which can be used to plot graphs, starting with linear graphs whose properties are then studied in their own right. Mappings, sequences and data tables might all be used to generalise the underlying as a 'position-to-term' rule. Most textbooks for younger students also provide input-output 'machines' to develop algebraic expressions of the rules of linear functions, which are variously called graphs, equations, rules etc. In our textbook searches we found very little consistency about when and how the term 'function' is introduced, although the non-statutory guidance suggests the word 'function' can be used in relation to input-output and mapping diagrams. In textbooks the word is sometimes delayed until formal treatment, and sometimes appearing without comment earlier on. Rate of change is first addressed implicitly as the gradient of linear functions, and this idea is extended to non-constant rates of change for quadratics with older students from 16 years (years 11/12) as a starting point for calculus through qualitative and quantitative study of increasing and decreasing functions. The phrase 'rate of change' was not salient in the English curriculum at the time of our study before early calculus, unlike in Israel[[2]](#footnote-2).

The typical English student will therefore not know there are mathematical objects called 'functions' until about 15 years of age, but will have some experience of linear graphs which are connected to input-output and mappings and may also be related to generalising linear sequences, and may have heard the word 'function' in relation to these. These will have been presented as with the *y* = notation. They will have met various graphs of realistic and statistical data, but these would not usually be described as functions. The only non-smooth, non-continuous, non-calculable graphs they will definitely have met would be journey graphs, but they may have met others through investigating other realistic phenomena. Graphs will have been constructed from calculating points and plotting, or from realistic data, and many teachers also use graph plotting software.

These generalisations provide a framework for analysing differences in Israeli and English students' responses to our task, and also allow us to make plausible conjectures about teacher input. All teachers in this study were 'functions aware' (Watson & Harel, 2013) meaning that they were knowledgeable themselves about functions up to first degree level and beyond, and hence could use their own knowledge to enrich students' experience beyond the curriculum and textbook expectations, so their students might know more than the typical descriptions above.

**Teachers' expectations**

In addition to the curriculum, we asked teachers who taught in the schools participating in our study what their expectations would be, since this question gives access into what they may have emphasised in their teaching.

The teachers from England said that students would meet input-output models in year 7 or before, and some teachers use the word 'function machines' to describe these. They said that explicit use of the word and its notation happens in their teaching of year 11 when studentsmeet transformations of linear and quadratic functions. The idea of mapping between sets is developed in year 12 as an introduction to a formal process treatment of functions in year 13. 'Rate of change' would first be discussed by these older students in the context of finding derivatives from first principles.

**Methods**

**Sample**

In England, the task (as part of a larger survey, see Ayalon, Lerman, & Watson, 2013, 2014a,b; Ayalon, Watson, & Lerman, 2015, first online) was given to two relatively high achieving classes from years 10 and 11 and also to first and the second years (12-13) of post-16, in which studying mathematics is optional. These years are equivalent in age to years 9 to 12 in the Israeli system. The classes were from two schools, each school providing data from alternate years. In Israel, the task was given to a middle achieving class from year 10, high achieving class from year 10, and high achieving classes from years 11 and 12. The classes were from three schools, with the two 10 year classes from the same school. Socio-economic backgrounds of all schools were similar relative to their national norms. The sample space was opportunistic, which is adequate for our research purposes which are exploratory, and hence we only used students who might be expected to proceed to further mathematical study which at some stage would include the need to understand what a function is. We use random anonymised samples of 10 scripts from each class. In this way we received 40 scripts from English classes and 40 scripts from Israel classes (total of 80 scripts). To make comparisons easier to read, we shall from this sentence on transform English year numbers to match Israeli, as these are more widely used internationally, and give the transformed numbers in square brackets. We will assume the middle-achieving Israeli year 10 group can be matched with an English high achieving year [9] group as the expectations of these groups are similar according to their national norms. We therefore have equivalent data from years 9 to 12 from students who are expected to do well in mathematics.

**Data analysis**

*Analysis of dominant ideas for functions*

Our analysis was qualitative, designed to identify characteristics of student responses that give insight into their understanding of functions. Numbers are used to point towards differences and phenomena that are important in terms of the function concept and that might be investigated further with larger samples appropriately matched.

Analysis of students' ideas of functions included two main stages. In the first stage, students' responses to the five fictitious students’ ideas of functions were classified according to the ideas of functions they expressed. With regard to each idea, a students' response was classified into one of the following four categories: (1) no indication (e.g., no answer) (2) misunderstanding of the given idea (i.e., incorrect choice and explanation), (3) partial understanding (i.e., partial choice backed with partial explanation or correct choice backed with incorrect or incomplete explanation), (4) full understanding (i.e., correct choice backed with correct explanation).

During this stage of analysis we found that interpretation of the statements could be more varied than we and the teachers anticipated, and also realized that most students tended to repeat the same ideas of functions when responding to different statements. This led to the emergence of a 'dominant' (most frequently-expressed) idea as a construct to access students' conceptualisations of functions, triggered by the statements. We also found that students' explanations for their choices were often more interesting than the choices themselves. Therefore, the second stage of the analysis, which we shall report here, included a holistic view of the set of responses from individual students, taking into account their five reactions to the given ideas and their response to the second part of the task (i.e., writing what is a function to her/him). This lead to characterising an individual's dominant idea, that is one that appeared most frequently in their response to the whole task, and other contributing ideas that occurred less frequently.

Decisions about interpretation were made collaboratively among the research team by discussion of generalities and possible ambiguities, and reaching a consensus. We constantly checked categorizations against the whole data set, and between ourselves, testing distinctions to see if all responses fitted into one, and only one, category and elaborating meanings and distinctions as we did so. and required several passes through the whole data and was informed deeply by our preceding work in this area (refs). The main problem was devising the method of analysis and ensuring that categorising was done solely on the written data and not on inferred meaning, thus we would expect other analysts to find the same results. This led to creation of extra categories where there were expressions that went beyond our previous knowledge of the field, and these decisions were justified within the team by looking at the data.

Seven ideas emerged either as dominant or contributing ideas. Five of them are those were presented in the task (input-output machine, mapping, relations between variables, covariation, and algebraic calculation). Two additional ideas were 'patterns in outputs' and 'domain'. 'Covariation', 'patterns in output' and 'domain' were found to be contributing ideas only. Below is a description of the categories, accompanied with examples from students' responses.

A response was coded as dominated by 'input-output' idea, when it emphasized a point-wise action of generating new values from given values, often using the language of input-output. Here are some examples of the kinds of responses that were coded as 'input-output':

1. Function is like a machine, that when we substitute a number, the value of x, we will get a number, the value of y. (Israeli year 10 student).
2. For me, a function is where a number, formula, or other means of numeric data is inputted, the function is applied to it to calculate, and then is outputted in the form of another number or formula etc. (English year [9] student).
3. Function is an input-output machine, when applying an algebraic equation to change one value to another. (English year [10] student).

Because of the explicit focus on algebraic calculation, the last two responses were categorized also as involving 'algebraic calculation' as a contributing idea to the dominant idea of input-output.

An 'input-output' response was categorized as including 'patterns in outputs' as a contributing idea when it attended to patterns that may exist in the numeric values of successive outputs. For example:

1. I think a function gives outputs to inputs, and the values of outputs show patterns. (English year [9] student).
2. The numbers show a pattern after being used in a function   
   (English year [9] student).

An 'input-output' response was categorized as including 'covariation' as a contributing idea when it attended to the manner in which the variables change together. Because half our total sample might not have met the words 'rate of change' we used 'change' in the statement, and it is possible that for some students this did not mean incremental or dynamic change, but merely using a different input value. We took a broad view of this, and decided that, if students expressed anything about how a change in input affects output, we would interpret that as covariation or a step towards a covariation understanding. For example, the following response refers to increasing and decreasing covering together:

1. I think functions are showing how inputs affect the output… the inputs always change the output, so functions will show how if the input increases, the output either increases or decreases (English year [11] student).

A response was coded as dominated by the 'mapping' idea when it emphasized a global process for connecting domain values to range values. Students referring to mapping usually emphasized in their writing the univalent idea to distinguish functions from other kinds of input-output or mapping process. For example:

1. A function is a one-to-one or many-to-one mapping. It links an element from the domain (set of possible inputs) to one, and only one, element from the range (set of possible outputs), and will always return the same answer for the same input (English year [12] student).

The above response was categorized as including input-output as a contributing idea as well.

A response was coded as dominated by the 'relations between variables' idea when it mentioned the relational aspect of functions and also dependency of variables. Here are some examples of the kinds of responses that were coded as 'relations' as the dominant idea:

1. Function represents the relations between the dependent variable and the independent variable (Israeli year 9 student).

The following example was coded as including 'covariation' as a contributing idea to relations between variables. The response focuses on the manner in which the variables might change together:

1. Functions are relations between variables and when there is a change in one variable, there is a change in the other variable, respectively… it can be either relations between variables from everyday life (like a function which matches students to their teacher) or trigonometric relations such as between numbers and their sinus, etc… however 'students - name of teacher' relation does not talk about how one variable changes in relation to another variable. (Israeli year 12 student).

Note that this limits the meaning of 'function' to a covariational model, whereas the following example was coded as including 'input-output' as a contributing idea to relations between variables, and would include the student-teacher mapping excluded by the previous statement:

1. Function is all about relations between variables. I see relations between *x* and *y* in functions when I put some input and it returns an output, and when I put a different number it will return me a different number as well, according to a certain regulation (Israeli year 9 student).

A response was coded as dominated by the 'algebraic calculation' idea when it associated functions with this idea exclusively. These responses usually showed no understanding of other given ideas of functions and often included an acknowledgment that the idea of functions as expression to calculate y-values from given x-values is familiar. For example:

1. I would guess she [Liz] is right as it sounds kind a familiar (English year [11] student).
2. This [Liz statement] makes sense. Finally I understand what someone is saying (English year [11] student).

A response was coded as including 'domain'' as a contributing idea when it referred to domain as something that has to be taken into account. Domain was used in these responses to provide restrictions to an idea for function (e.g., input-output or algebraic calculation). For example:

1. In functions that have a restricted domain not every input will give an output (Israel year 10 student).
2. For every function that we will take and insert to it an input, we will receive an adequate output, except for functions with limitations like *y*= ; x=1, if we insert 1 we will get a meaningless expression, and this is not an output (Israeli year 12 student).

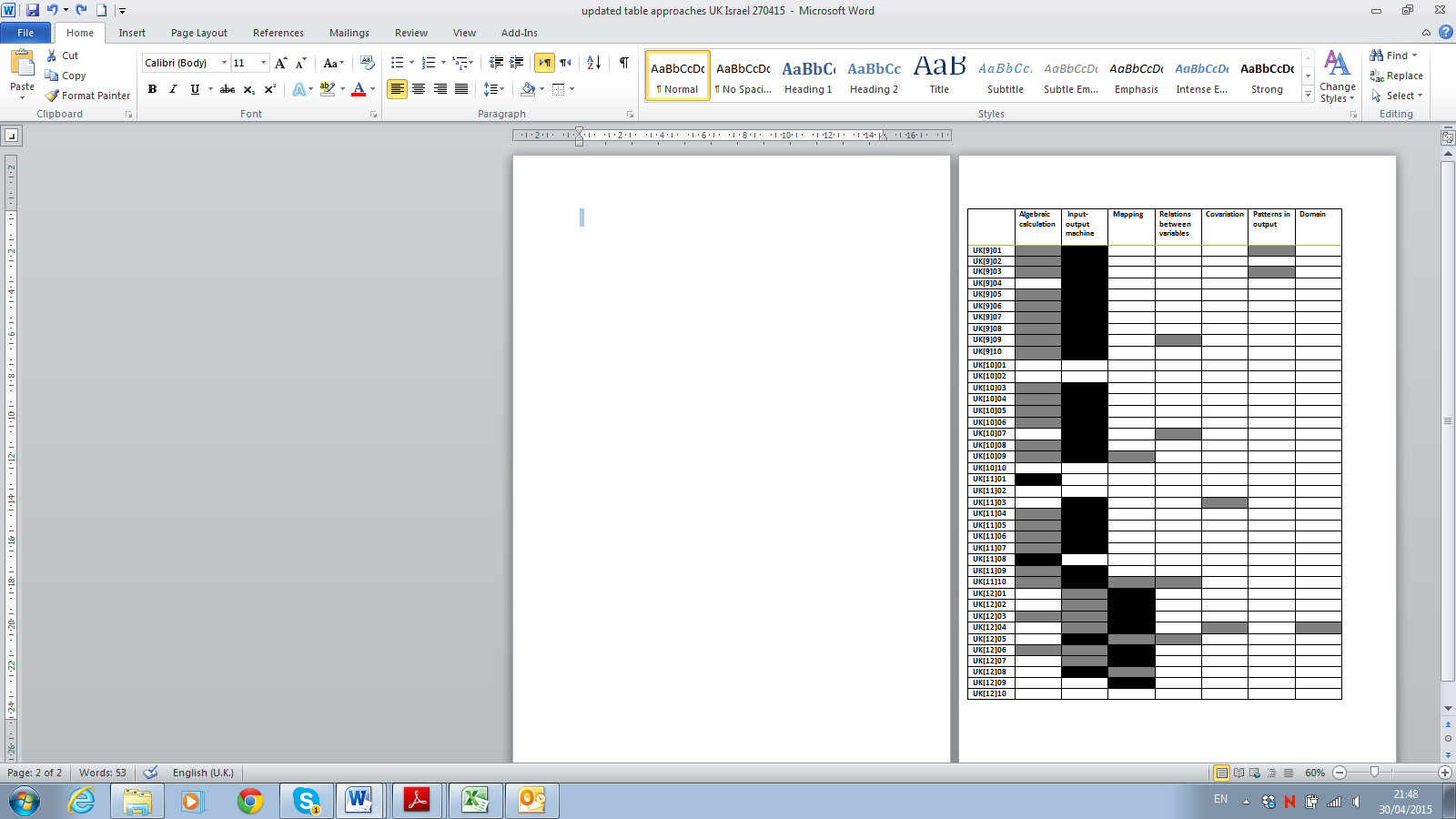
*Analysis of example spaces*

We did not ask students to provide examples, nevertheless the extent of example use in students' responses led us to analyse these in terms of personal example spaces (PES) (Watson & Mason, 2005; Sinclair et al., 2011). Students' own choice of using examples to explain their answers was particularly apparent when they wished to disagree with the statement by showing counterexamples or restrictions, e.g. on the domain, or extensions of meaning. Example use was particularly important for some students who did not express clear meanings in their words.

The total of 79 examples fell into the following 8 categories:

1. linear functions;
2. polynomial non-linear functions;
3. non-calculated functions whose values are not achieved by calculation at school (e.g., trigonometric, exponential);
4. constant function;
5. functions with restricted domain;
6. functions with more than two variables;
7. inverse functions;
8. non-mathematical functions.

**Results**

Figure 1 and Figure 2 present the distribution of approaches to functions as revealed in the task, within the English and the Israeli students, respectively.

Black cell: Dominated idea; Grey cell: Contributing idea

Fig. 1. English students' approaches to functions

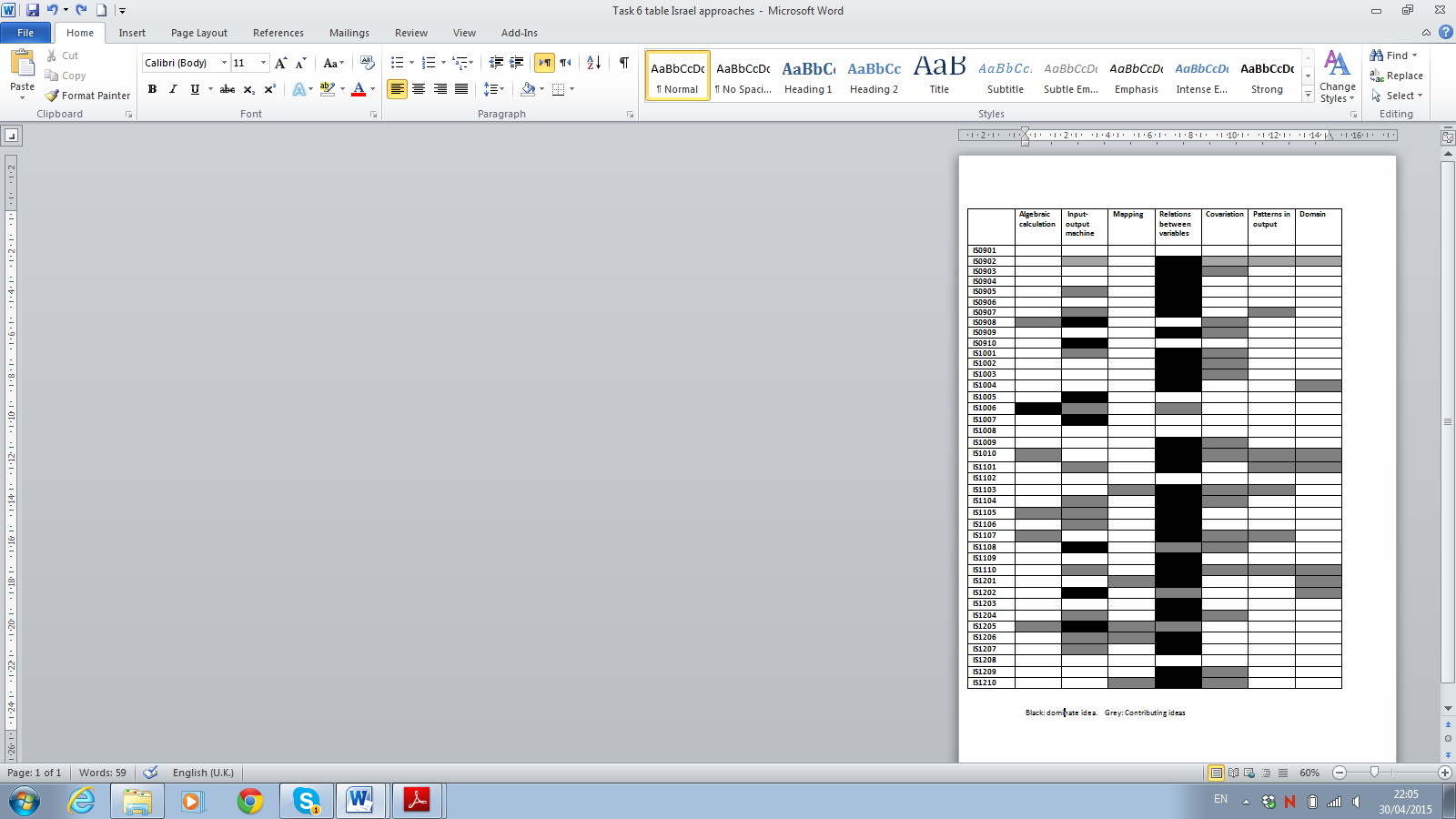
Black cell: Dominated idea; Grey cell: Contributing idea

Fig. 2. Israeli students' approaches to functions

Five English students (out of 40) and four Israeli students (out of 40) were categorized as "no indication for approach to functions" (appear as blank rows in Figures 1 and 2). In the following we refer to the other 35 English and 36 Israeli responses.

As shown in Figure 1, most English students (26 out of 35) had a dominant view of functions as input-output models. Seven other students, all in year [12], expressed a dominant view of functions as mappings. Two students focused strongly on algebraic calculations.

For many of the English students dominated by the input-output view (17 out of 26), the idea of functions as algebraic calculations was voiced as well, while other ideas were not. These students focused on the computational aspects associated with functions (examples of students' responses are included in the analysis section). For few other students, the idea of input-output was accompanied by one of the other ideas, such as patterns in outputs or relations between variables.

One-to-one or many-to-one mapping dominated year [12] English students' communications (7 out of 10 students). These students emphasize in their writing the univalent idea which distinguishes functions from other kinds of input-output or mapping process, as well as expressing a more global approach for connecting the two sets.

In contrast to the English students whose dominant idea of functions was of input-output, most Israeli students (28 out of 36) had a dominant view of functions as relationsbetween variables (see Figure 2). This idea dominated their thinking all through all years. The covariation and the input-output views appeared as contributing ideas among 17 and 13 students respectively out of 36, spreading throughout years 9 to 12. Four other views were also understood but by fewer students.

As also shown in Figure 2, most of the Israeli students presented flexibility in interpreting the statements given in the task. Three students showed a rather restricted view of functions as input-output machines exclusively.

**Example spaces**

We now describe the type of examples that students used to explain their responses as these give some indication of their personal example spaces. We also look at how these are related to the different views of functions presented in the task. This information gives a window on how the curriculum has been taught and how the students' concept images have been built (Watson & Mason, 2005; Sinclair et al., 2011). Note that we did not explicitly ask for examples, so students have also made their own choice to use examples to explain their answers.

Table 1 presents the presence of the examples of functions offered by the English and the Israeli students.

Table 1

*Examples of functions offered by students*

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Linear functions | Polynomial non-linear functions | Non-calculated functions | Constant function | functions with restricted domain | functions with more than two variables | Inverse functions | Non-math functions | **Total** |
| English | 15 | 0 | 1 | 3 | 0 | 1 | 3 | 2 | **25** |
| Israel | 5 | 4 | 9 | 14 | 7 | 2 | 3 | 9 | **53** |

As shown in Table 1, the Israeli students gave 53 examples, many more than the English students who gave 25. Examples of functions that are not of the form *y = mx + c (m ≠ 0)* were prevalent in the Israeli data (48/53). In contrast, out of the 25 examples given by the English students, 15 were of linear functions with m ≠ 0.

In the Israeli data the existence of constant functions was mentioned 14 times; 9 of these occurrences were to argue against Naomi's statement that functions show how one variable changes in relation to another variable, therefore not admitting a zero rate of change. For example:

Not always the function shows how one variable changes in relation to another variable. There are constant functions in which the *x* variable varies but there is no change in the y variable, such as *y*=3. (Israeli year 11 student).

The other 5 occurrences of constant functions were to indicate that a function might not be one-to-one mapping, but many-to-one mapping. For example:

There are functions that are not mapping one element of one set to one element in a second set, but many elements to one element, like the constant functions, for example *y*=3. (Israeli year 11 student).

There were 9 mentions of functions that cannot be calculated algebraically, such as exponential, root and trigonometric functions. These examples were mainly proposed as counterexamples for Liz statement of functions as expressions to calculate *y*-values from given *x*-values, suggesting that there are other types of functions which cannot be simply calculated. For example: "Because there are functions that do geometrical operations like trigonometric functions" (Israeli 10 student).

9 mentions of nonmathematical functions appeared, most of them were used to argue against a covariation view of functions. For example: "Matching students with the name of their teacher does not talk about change in variables", or "Relation such as between a person and the political party he votes for does not express how these variables change together" (both of Israeli year 9 students). Several other non-mathematical examples were offered by students. For example, as a response to our question about the meaning of functions, a 10 year student wrote:

A function is a certain action that you put an input and it returns output. They can be found in many domains and the mathematics only shows its theoretical side. Functions are the spirit beyond almost every activity in the world, washing machine, responsible for most actions of the computers and also human beings who are type of computers. Functions exist in our DNA. One of the most important issues in math.

There were 7 mentions of functions for which the domain has to exclude certain values so that zero does not appear in the denominator of rational function. One student called these 'non-continuous functions'. These examples were mainly presented as restriction to Arthur's statement of functions as input-output machines, explicating that the input cannot always be any number. For example:

There is a domain and the function cannot receive every *x*. For example : *x* cannot be 0 (Israeli year 9 student).

The quadratic and other polynomial functions were mentioned 4 times to indicate that a function might not be one-to-one mapping (Ruth's statement). For example: "There are functions which give the same output and there is no 1-1 matching. For example *f*(*x*) = *x*2 for *x*=1 and *x*=-1 there is the same *y*" (Israeli year 12 student). Three students argued against a calculation view because it was expressed in terms of *y*=*f*(*x*) and they wanted to say that this could equally be expressed as *x*=*f*(*y*). Two students suggested functions that may have more than two variables to argue against a covariation view.

The English students' exemplification was much more limited in scope, consisting of objections to the covariation statement using: non-mathematical relations (1 mention); constant functions (3 mentions); inverse functions (3 mentions to show that variation could be seen from the point of view of *y* as well as from *x*); and functions which have more than two variables (1 mention). The square root was offered by one person as a counterexample to a calculation approach, as it offers 'multiple outputs'. There was one suggestion also of nonmathematical relations offered as a challenge to the numerical input-output perspective.

Where the Israeli students gave many more examples of contradictions, extensions and restrictions, as described above, the opposite was true when students were offering examples to illustrate their agreement or understanding. English students offered 15 examples to show their understanding of linear functions as algebraic calculations or input output actions, either to support the relevant statements or to give their understanding of functions. Israeli students offered only 5 agreement examples, 4 of which were to illustrate algebraic calculations.

**Summary and Discussion**

Our summary is not intended to be a claim about these students' whole knowledge about functions, because all we know is their responses to our tasks, and some of the background that has contributed to their concept images and responses. Nor can we generalize about countries on the basis of a small sample. We claim to give a window into their understanding and suggest influences. We do this through using comparison as a mechanism to make conjectures about how understanding of the function concept can grow according to curriculum conditions, although we are mindful of how the work of individual teachers and the available textbooks and assessment systems mediate these. This is why teachers' expectations have also been taken into account throughout our comments so far.

Most Israeli students in this study had a dominant view of functions as relations between variables, and this view dominated their thinking before the advanced treatment of functions in the final secondary years and was sustained until year 12. The covariation view (according to our broad meaning) and the input-output view were also expressed by many of these students, spread throughout years 9 to 12, showing flexibility in interpreting the statements given in the task. Other views were also articulated but by fewer students. Teachers had not expected this flexibility of understanding, and in particular had not expected the relatively copious use of counter-examples to argue against the covariation and algebraic statements, and this use also contrasts with studies in which the provision of counter-examples was difficult (Zaslavsky & Ron, 1998). We conjecture that in our study students already had a supply of potential counter-examples available from the teaching they had experienced. Also, the logical situation in which they were supplying these was more like non-examples ('here is a function which is not in the given subset') rather than counter-examples to an argument.

As the teachers anticipated, most English students in this study had a dominant view of functions as input-output models, with year [12] showing a shift towards a mapping view, while retaining also the input-output view. Another popular view was of algebraic calculations. Other views were understood by very few students, and in particular the understanding of the covariation view was only shown by two students in the context of the word 'function'.

The shift from seeing functions as action to functions as process when those functions are algebraic is important (Dubinsky, 1991) but difficult to discern from the students' writing. While it seems credible that a function machine model can form a bridge between actions and process conceptualizations, we cannot assume this. Some students associate input-output with algebraic calculations and we assume this is an action view that accords with their experience,but we cannot assume that the other input-output responses are associated with process conceptualization although it is likely that some of them do. For this reason we have not made analytical claims about the action/process dimension.

Israeli students used a variety of examples of functions to explain their responses; several of them (e.g., constant functions; uncalculated functions) were reported in the literature as being difficult for students to understand as legitimate examples of functions (Leinhardt et al., 1990; Markovits et al., 1986). Compared to the Israeli students, the English students' exemplification was much more limited in scope. The difference in example provision, as well as indicating limited experience (Watson & Mason, 2005), could also be an indication of the relative use of examples in teaching, so that these Israeli students might be more habituated to giving counterexamples to refute while English students give examples to illustrate understanding (Watson & Mason, 2005). It could reflect the English pedagogic focus on informal assessment for which giving examples is a common tactic used by teachers to evaluate understanding across the whole class; or it could reflect that only well-behaved algebraic functions are named as such initially, so they have no counter-examples. We cannot say that English students did not understand the logic of counter-examples; it is more likely that they had none available from their experience.

It seems as if the Israeli students considered a variety of views, both correspondence and covariation, under the umbrella of an association of functions with relations between variables. The English students mainly only considered a point-wise view of functions (Even, 1990), shifting to a correspondence view by year [12]. However, functions are not introduced by name to them until years [10/11] apart from being used as 'function machines', so we conjecture that whereas Israeli students had many experiences to which the word 'function' was attached, English students may have had similar experiences, but without the unifying word 'function'. The repertoire available to younger Israeli students, and their willingness to draw on a personal example space for counter examples, supports our conjecture about their richer example spaces being the result of the different curriculum. Students were able to build a varied example space when the label of 'functions' was used continually, providing a concept space that can be gradually populated with successively more sophisticated examples accreting around the word.

Indeed, our analysis of other tasks (Ayalon, Lerman, & Watson, 2014a,b; Ayalon, Watson, & Lerman, first online) shows that 33 of these 40 English students had been able to handle covariation as relative change in variables successfully in a sequential task, and/or as a qualitative sense of instantaneous rate of change in contextual tasks, so it can be assumed that the word 'function' did not trigger these existing capabilities for them. Israeli students, having had a general introduction to functions of many kinds in year 7 had subsequently made more connections between the word and its various possible meanings. However, there was evidence of English students trying to fit their knowledge of sequences into the task in this study: 5 talked about the function being the *n*th term of a sequence, and 2 described patterns in output being produced by functions, thus trying to connect sequences of output values with a process understanding of functions. This adaptability among English students was also seen in our analysis of another task, in which they had to adapt correspondence and/or covariation approaches to a task in an unfamiliar form. (Ayalon, Lerman, & Watson, 2014a; Ayalon, Watson, & Lerman, first online). 37 of the present sample made suitable adaptations and 29 of these were successful.

Evidence for an object view of functions (Sfard, 1991) is mostly implicit, in that students were responding to the word as if it was a noun describing something, which could have been a set of actions, a process, or an object that had behavior and purpose. In that sense we 'forced' an object view by the grammar of the statements. A few students wrote that they did not understand, or tried to fit the word to knowledge through its everyday meaning, but most made a connection with the accepted dimensions of functions knowledge (e.g., those provided by Leinhardt et al. (1990), Tall (1992), and Ronda (2009)). Our data, therefore, provides no insights into who has or has not an object view, merely that most students could respond to an object use of the word. Our data does show, however, that students have different 'object views' and that the treatment of the idea of function as a noun is not enough to guarantee a full range of meanings – becoming an object and being fully understood could be separate lines of development.

This has implications for any discussion of whether it makes more sense to name mathematical ideas formally after students have experienced them, or whether to name them before. This is not a question of whether formal treatment has to precede exploration and elaboration, but of whether and how students are supported to make connections, through language, between various mathematical ideas. In particular, it shows how different concept images might be constructed by different curricula, the conceptually limiting effects of limited example spaces, the habituating effects of example and counter example construction in classrooms, and how adaptability can occasionally triumph over limitations of experience.

Turning to the psychological literature on word, meaning, sense and thought, picking up on our earlier comments on the significance for us of Vygotsky’s work in particular, it is clear that he struggled with the relationship between these. According to Zinchenko (2007), Vygotsky dictated the last chapter of *Thinking and Speech* (1986) as he lay dying. His formulation in that last chapter differs greatly from his first chapter in relation to meaning and sense and the word, but of course there was no way he could rethink and rewrite the whole text. Zinchenko says that at first Vygotsky had a fairly weak role for meaning: “as a meditational means between thought and word” (*op cit*, p. 226). At the end of the book he indicates that meaning is the public side of word whereas *sense* is “closer to the dark side”, in that it is private, though expressible if desired.

What is clear, though, is that Vygotsky retained the idea that word is the beginning of cognition, in the same sense that pointing to or pointing out an object begins the development of sense and meaning. We might conjecture, then, that having a name, a label, in this case *function*, early in their learning, around which concept images are built over time is a support for students in their learning and may well be critical in the early development of the more coherent structure of understanding as seen in general among the Israeli students than the English. The word is the beginning of cognition, which then calls for all the work of the teacher and the resources to bring meaning, through concept images and activities, building towards sense. We need to be clear, though, that we are not advocating that in teaching mathematics we should return to starting with title and definition as if these ‘naturally’ encapsulate a concept, nor are we saying that the Israel curriculum is, in any sense, 'better'.

In terms of the issues raised in this study, there is some research that can be done to compare systematically the conceptual learning effects of curricula that differ in the attention paid to the deliberate use of the word 'function' as an organiser in a range of mathematical contexts, and it could be used with or without formal treatment.

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1. A more recent revision of the English curriculum is described in groups of years: years 7 to 9, years 10 to 11, and years 12 to 13. Content and order of relevant concepts has not changed significantly from what is described in this paper. [↑](#footnote-ref-1)
2. This may change with the new curriculum [↑](#footnote-ref-2)