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4. pedagogy of variations

Synthesis of various notions of variation pedagogy

Abstract

Using examples of practice from past textbooks and current research I explore how variation theory is manifested in mathematics teaching, and hence identify ways in which the variations embedded in mathematics can be made available for learners by teachers. Furthermore, I identify the notion of the 'dependency relationships' that are invariant in mathematics and which are often the intended object of learning. I look at various ways in which people draw attention to these invariants through use of variation and compile statements that expand variation theory within mathematics education. Finally, I illustrate how careful use of variation can lead to abstraction of new ideas.

Keywords

Dependency relationship; variation theory and abstraction; traditions in variation theory in mathematics; inductive reasoning; mathematics pedagogy

Introduction

In the last decade there has been a gradual increase in publications about the potential of variation theory for mathematics education. Mathematical concepts are often encountered by learners through examples, and the variation they experience through examples that have some similarity in structure leads them to generalise either about the properties of mathematical objects or about relations between them (Michener, 1978). The word 'variation' therefore elicits consideration of the possible variables that can be manipulated in teaching mathematics and designing tasks.

Task design always has with it, either explicitly or implicitly, assumptions about pedagogy, a fundamental belief being that learners will notice and generalise from patterns and relationships between what aspects vary and what aspects are invariant (e.g. Mason, 2000). By using historical examples, and examples presented at the 2013 ICMI Study Conference on Task Design in Mathematics Education (Margolinas, 2013), I reflect on the contribution that variation theory makes to our understanding of pedagogy in mathematics education. Through these examples I explore how variation is manifested in mathematics teaching, and identify a notion of 'dependency relationships' that are invariant in mathematics and which are often the intended object of learning. I look at various ways in which people draw attention to these invariants through use of variation and I compile statements that expand variation theory within mathematics education. Finally, I illustrate how careful use of variation can lead beyond generalisation to abstraction of new ideas. The ICMI Study provides a snapshot of current practice in task design, and included a panel of presentations about the current use of variation to design tasks and pedagogy. However, I shall start with some observations from earlier examples of pedagogy.

mathematics pedagogy of variation from some past textbooks

I start with the beginning of a set of questions from a typical algebra textbook published in English in the early 20th century, chosen at random from a collection (Paterson, 1911, p.120):

|  |  |  |  |
| --- | --- | --- | --- |
| *1. (2x)3* | *2. (-2x)3* | *3. (-2x)6* | *4. (2x)6* |
| *5. (2x2)3* | *6. (-2x2)3* | *7. (-2x2)6* | *8. (2x2)6* |

The exercise goes on with more variation of letters, powers and structures, but these first eight questions provide raw material for more than mere practice of using algebraic notation. The examples have been constructed with care to draw attention to what varies and what stays the same, and to possible misconceptions a student may have, or mistakes they may make. The author goes on to say:

The student will now have found out the following rules: (i) an even power of a negative quantity is positive, an odd power of a negative quantity is negative; (ii) *(xp) q= xpq ...* (p.120)

This is followed by several other 'rules' which were to be 'found out' by reasoning inductively from the answers to the exercise. Paterson is explicit about the assumptions of his task construction: namely that learners will be led experimentally to a difficulty that has to be overcome, such as meeting '3 - 5' before negative numbers are formally introduced (p. A2) and noticing the difference between this and other examples. This visually obvious use of variation is not replicated very often in his book. More usually, Paterson uses varied collections of questions in which one subsequence of questions holds a particular similarity that is then disrupted in a new subsequence. Both of these uses of variation - with and without visual similarity - are used in most algebra textbooks of that era.

Knowledge of the value of careful variation also appears in some mid-twentieth century texts produced in England by teacher teams. The Midlands Mathematics Experiment, which was a syllabus based on New Mathematics, introduces straight lines by varying *L1 = {x,y: y=x}* to give:

|  |  |
| --- | --- |
| *L2 = {x,y: y=x + 1}* | *L4 = {x,y: y=x - 1}* |
| *L3 = {x,y: y=x + 2}* | *L5 = {x,y: y=x - 2}* |

and then asks: 'What do you notice about the lines *L1, L2, L3, L4, L5*?' (MME, 1970, p.53). Again, the purpose is to provide variation against an invariant background and the implied pedagogical intention is that learners will conjecture about the relationship between the variations in the algebraic expression and the differences in the graphical depiction of the sets of points. In this case, the chosen set of lines implies an understanding of the effects of near-simultaneous variation between examples but, unlike Paterson's era, such use of variation, whether visually obvious or not, is not easy to find in textbooks during the later part of the 20th century, nor currently.

Both of the above textbooks assume that teachers will encourage appropriate reflection in learners, and not merely rush on to the next idea or technique and the next set of questions to do. Variation, in the examples I have given, is not random but relates closely to the concepts or conventions being introduced. In each of the examples above, variation is also indicated visually by using similar notations so that the difference in outcomes can be related to the visual difference as well as to differences in mathematical meaning. However, other exercises from Paterson's book have an underlying attention to variations in structure and complexity *without* a visually invariant background. For example (p.125):

|  |  |  |
| --- | --- | --- |
| Find the square of each of the following expressions | | |
| *a + b - c* | *x - y - z* | *- x - y - z* |
| *2a + 3b + 1* | *2x + 3y - 4z* | *3y - 4z + 2x* |

Obviously there are opportunities here for an astute teacher or student to make comparisons, but the use of *a, b, c* and *x, y, z* among the examples introduces a further variation that, while necessary for becoming fluent and competent with algebra, makes comparison across all examples less likely without teacher intervention. Marton would describe this as 'fusion' (2015, p.51) in which a learner has to appreciate the underlying mathematical idea (squaring trinomials) by fusing variations in notation, in coefficients and in signs so that they can recognise the idea whatever the combination of letters, numbers and signs. Such variations might have been met separately and are then brought together, fused. We could argue that if every mathematical idea were to be broken down so that learners had to work on each possible dimension of variation separately the whole business of learning mathematics would take decades. Nevertheless a reliance on generalising from varied examples is evident in some early algebra textbooks. For example Godfrey and Siddons (1915, p.vi) suggest 'rough induction' as a 'sound' way to approach algebraic laws.

Current 'traditions' in variation theory

Having shown that variation was used purposefully in mathematics pedagogy before recent theoretical articulations, I now present two examples from the 2013 ICMI Task Design Study Conference to illustrate some current thinking. These use the two main approaches to variation theory in the literature that are now well-established, a Swedish approach represented in this paper by Kullberg (2013) and a Chinese approach represented in this paper by Sun (2013).

The Swedish 'tradition'

The Swedish approach originated in phenomenographic analysis of learning, which enables the researcher/developer/teacher to identify a critical aspect of the concept being learnt. These aspects are found in the variety of students' responses, rather than in analysis of the concept. The theory propounded by Marton (e.g. Marton and Pang, 2004; Marton 2015) is that unless this critical aspect is varied against an invariant background, students will not notice it, so therefore its variation becomes a design imperative for teachers. However, application of this idea in classroom settings suggests that students experience variation not only through the task as written, but also through pedagogical acts that draw attention to the variation (Al-Murani, 2006; Kullberg, Runesson, & Måtensson, 2013).

Three full learning study cycles were carried out collaboratively with a small group of teachers by Kullberg and her colleagues (Kullberg et al., 2013). The part of the study I am going to look at focuses narrowly on the critical aspect of division that when the denominator, or divisor, is smaller than one (but still positive) the quotient will be greater than the dividend. This is the intended object of learning. For some understandings of division this idea is counterintuitive, since division in everyday life tends to involve reducing things in size, or sharing quantities between people. In this study the 'critical aspect' for division was chosen from the teachers' knowledge of students' difficulties. However, this same critical aspect could have been chosen as a result of analysing students' work on a particular task, or from clinical psychological research into children's understanding. The proposed variation is therefore the divisor, and its effect on the quotient, to draw attention to what happens when it is smaller than one by contrasting with what happens when it is greater than one.

Learning is described as 'seeing something in a new way by experiencing aspects that you have not experienced previously' (Kullberg et al., 2013, p.616) and this is achieved by varying a critical aspect of the object of learning against a background of invariance thus producing something to notice - in this case the behaviour of a relationship. The teachers therefore co-planned the following collections of examples that would be used in the lesson (see figure 1).

|  |  |
| --- | --- |
| 100•20 = 2000 |  |
| 100•4 = 400 |  |
| 100•2 = 200 |  |
| 100•1 = 100 |  |
| 100•0.5= 50 |  |
| 100•0.1= 25 |  |

Figure 1: Examples used in a lesson on division (adapted from Kullberg et al.,2013, p.617)

I have subsequently presented this collection to a range of teacher or student audiences to understand reactions to the visual layout. The first thing that most people notice, through visual perception, is that 'there are lots of hundreds' or every line starts with 100. A little while later they may say that along each line there is a fraction with 100 at the top. The first column of equations shows multiples, and answers get smaller; the second column shows divisions, and answers get larger. The reason these similarities and differences are easily noticeable is because of the layout, so if you were responding visually to the spatial structure you could say that there were similarities in the lines and columns of the layout. These similarities match exactly the abstract mathematical similarities that are instantiated by the numbers and represented by the numerical symbols. By this I mean that extending the sequences of equations downwards or upwards using superficial similarities and number sequences, the horizontal relationship between multiplication and division would also be preserved. The form and structure of the visual representation matches the form and structure of the underlying mathematics. Mathematically, the division examples aim to hold the dividend (100) invariant while varying the divisor so that the intended object of learning appears as the varied behaviour of the division. When the intended object of learning is the behaviour of a dependency relationship, as it is here, there has to be variation in at least two features: at least one input feature and, hence, the output feature. The relationship itself has to be inferred through discerning the effects of its behaviour, so showing the possible differences in behaviour is clearly a sensible thing to do. Kullberg et al. (2013) refer to this as using 'contrast' after the four possible patterns of variation*:* *contrast, generalisation, fusion, and separation* put forward by Marton, Runesson, & Tsui (2004).

Variation theory, as well as being used at the design phase, can also be used to analyse what is made available to be learnt through the pedagogy that accompanies the designed task. The lessons from this shared plan were analysed by looking for variations in the way the plan was used, that is the ways in which teachers drew learners' attention to the embedded relationships. The study shows how many aspects of the learning environment, such as teachers' and students' verbal interactions and gestures made a difference to what was available to be learnt by drawing attention to the possible relationships between elements in the examples. The authors drew mappings to show how different teachers pointed to different connections in the examples (Figure 2).

Figure 2: Teachers' gestures to explain examples (from Kullberg et l.,, 2013, p. 618 ff.)

In the partial version on the left the teacher is connecting the divisor to the quotient, which could emphasise that as the divisor gets smaller the quotient gets bigger; however, it is also possible to focus on the fact that their product is always 100, which is what the teacher talked about. In the example of the right the teacher is less systematic about the relationship between multiplication and division, but points instead to illustrations of various aspects of the fundamental relationship, contrasting an example with divisor>1 with one where divisor<1, and was also talking about this contrast. This is important for the way that variation theory is seen within the mathematics education field, because a superficial look at it might suggest that the theory as applied to design is purely cognitive and concerned only with what is presented to learners in order for them to construct meaning individually through inductive reasoning. The addition of teachers' gestures and speech to the mix indicates a need to think also about attention and the disposition of learners to discern what is intended.

Kullberg's use of variation in this paper focuses only on what is *available* to be learnt, but we can surmise what students focus on. It could be argued that the same (not varied) gestures in different cases might help children learn about essential structure because it provides another representation; repetitive action, gesture, rhythm and speech is seen by some authors as an important aspect of learning structured knowledge (Hewitt, 2006). The gestures indicated in the leftmost example, together with the speech, emphasised the complementarity of division and multiplication through invariant gestures; the rightmost example gestures selected two contrasting examples that, with the speech, drew attention to the intended object of learning.

From this study we can see the value of varying the critical aspect, so it can be noticed by learners, and also the value of layout, speech and gestures that draw attention to the background invariant structure. In some of the textbook examples above, the 'drawing attention to' was intended through reflective questions and comments by the authors.

The Chinese 'tradition'

I now exemplify the Chinese tradition of focusing on change. Sun (2013) points to two key features of task design: OPMS (one problem multiple methods of solution) and OPMC (one problem multiple changes). My interpretation of 'multiple changes' in the examples that she gives is that these can be transformations of layout and different representations. Also at the bottom of the page the numbers are changed, but the underlying structure of the problems is the same, that is similarly structured and hence similarly cognitive. Comparison and connection of these varying experiences is understood to be a fundamental mental activity. Sun uses variation as an analytical lens to describe the way that the official Chinese textbook presents the additive relationship as an object of learning (see also Marton 2015, p.245).

The page of the Chinese textbook (Figure 3) first shows four symbolic transformations of the relationship between 10, 3, and 13. It also shows physical materials that can be used, and some number facts between 11, 2, and 13 that can then be derived. It draws attention to symbolic notation by offering varied transformations of the relationships in which certain numbers have to be inserted. This illustrates two uses of OPMC. Firstly, Sun shows 'one problem with multiple representations' with each representation leading to a different solution method (OPMS). Secondly, she shows 'one problem with different parameters' which require the same solution methods as above, but instantiated with different numbers. Contrasting this approach to the Swedish approach, we can ask: is the learner supposed to learn about the representations, because those are the things that vary? and: is the learner supposed to learn about the different solution methods because those are the things that vary? The variation is being used to support a variety of ways in which learners can enact and record the basic part-part-whole relationship, the additive relationship, which includes addition and subtraction as the relationship between 10, 3, and 13. Rather than learning about one varying critical aspect, the learner has to learn about the relationship. Is the learner supposed to learn only about 10+3 equals 13, because that is the thing that does not change? Well not that either, even though this specific relationship is the invariant. The underlying principles are more complex and sustained than these questions would imply.

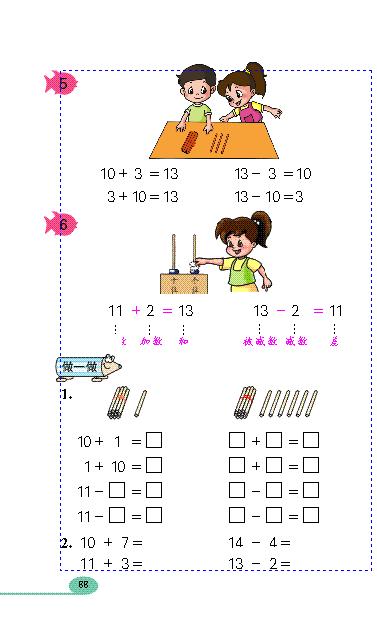


Figure 3: Page from Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005)

On the page the actions and the notations are specific enactments of a fundamental relationship which will appear again and again throughout children's early learning of number. Deliberate and planned variation is obvious, but the core idea - the relationship between addition and subtraction - is constant throughout several pages of the book. It is the *invariant* nature of the additive *relationship* that is the intended object of learning, together with the varying ways in which we might recognise it. Sun describes and demonstrates that the space of learning is about blending patterns of variation and invariance by juxtaposing problems, examples, illustrations that resemble each other in some respects but differ in others so that learners have something to discern.

My understanding of what is presented by both Kullberg and Sun is that the variations presented visually in mathematics give a direct access to important variations around a core conceptual idea: in the example I have given from Kullberg, the visual layout can be used to draw attention to the mathematical structure that causes the variations in output, as well as drawing attention to the variation in outputs themselves; in the example I have given from Sun, the visual layout also gives access to actions with materials that embody various perspectives on the mathematical structure, and notations that record the relations embedded in those actions. In Sun's case, the intended object of learning is an invariant relation, whereas in Kullberg's case, the intended object of learning is a particular varying feature of the underlying relation.

However, mathematics is not merely the product of inductive reasoning from examples, so the provision of carefully varied examples and hoping that, or directing attention so that, learners may generalise inductively from experience cannot provide a full mathematical learning experience. In Sun's work, it is not only inductive generalisation that leads to insight, nor is it only contrast and fusion (to use Marton's words) but some kind of cognitive work that abstracts the additive relationship from those processes. The cognitive work involved in using the experience of working with examples to identify implicit relationships is described by Gu, Huang and Marton as 'conceptual' and by Leung, Baccaglini-Frank & Mariotti (2013) as the identification of *level-2 invariants*, that is invariants that have to be discerned through experiencing an invariant relationship underlying variation. Cai & Nie (2007) talk about MPOS, multiple problems with one common method of solution, as also indicating that different problems might be manifestations of the same dependency relationship.

Learners have to be able to discern, to read, what is presented visually as a collection of parts - what Marton calls 'separation' (2015, p.53) and we might also call 'analysis' (Huang, Mok and Leung, 2006). The more mathematics we know, the more possible ways of reading a page of mathematics, or one mathematical expression, we have. We can go beyond the visual similarities and differences, especially with pedagogic help and direction. We can also go beyond induction, which enables us to build conjectures about objects and relations we have not yet met; but the more mathematics we know, the more capable we are of *de*duction by seeing a particular example as an instance of something we know about more generally, and being able to reason about its properties. Many textbooks tend to confuse these by offering collections of very varied examples from which little can be either induced or deduced, the purpose being to rehearse procedures. Difficulty is ramped by varying numbers, signs and arrangements rather than by scaffolding progress towards conceptual understanding, as is indicated in the Chinese example.

Variation in dynamic geometry

Having set up a contrast between learning the feature that is varied, and discerning an underlying invariant relationship, I now move to a geometrical example presented at the ICMI Study from a body of work by Leung and Lee (2008). In geometry we would expect visual representation to relate very closely to the relationships being represented and Leung has done extensive work to connect dynamic actions to underlying relationships. Tasks are presented in a dynamic geometry environment. This allows students to use the action of 'dragging' to vary particular features of a diagram while shifting their attention between parts and the whole of the object of exploration, consistent with a perspective of discernment of a concept through variation. The invariant patterns of a configuration can then be 'separated-out' by observing patterns of change.

It is this kind of use of controlled variation in mathematics that initially attracted me to variation theory, since it provides a connection between structures inherent in mathematics, the components of the learning environment, and plausible variation in students' learning.

In several of Leung's studies a digital record is kept of the work undertaken by a large number of students in a geometrical context (e.g. Leung, 2008, 2011, 2013). In the example I use here (Leung and Lee, 2008) students have been asked to vary a given quadrilateral ABCD by dragging point D so that the quadrilateral then has at least one pair of parallel sides. There are several possible correct answers which would be achieved by applying full geometrical reasoning, and a digital record of variation among students' answers provides an instant 'phenomenograph ' of the outcomes (Figure 4). From this picture it can be deduced what subset of appropriate geometrical properties is being used by students.



Figure 4: Graphical record of students' 'dragging' solutions (from Leung & Lee, 2008)

What has happened is that students have digitally dragged point D and left it somewhere new. The clusters of points show two distinct patterns: one pattern makes AD parallel to BC and the other pattern makes CD parallel to AB. Where these cross there is a further cluster that makes both pairs of lines parallel. There are also several points that show possible misunderstandings or alternative interpretations of the task. Thus the digital record gives a window into the relationships between the diagram, the task and their knowledge and capabilities. We observe this by, in Leung's words, 'strategically *contrasting* and *comparing*, *separating* out critical features, *shifting focus of attention* and *varying features together* to see whether invariant patterns emerge' (2013, p.7). Dynamic digital software makes it possible to use what he calls a 'sieving' principle, that is to sieve out and display certain critical invariant features of a dependency relationship.

Direct perception is used in this task in two ways: firstly for the student who is trying to enact their understanding of 'parallel' and secondly by the researcher who is trying to map the range of possible understandings. Both of these perceptual acts lead to classical geometrical properties which can be seen and described visually, for example the properties of a parallelogram where both pairs of opposite sides are parallel. Leung's use of the phrase 'invariant patterns' is not referring to optical patterns, but to dependency relationships as level-2 invariants that are manifested through the behaviour of variables. By contrast the examples from Kullberg and Sun are both about mathematical properties that cannot be seen directly, but only experienced as the invariant connection between features of varying examples.

Although Leung and Lee's work described here is concerned with finding out what students do, a key aspect of good mathematics pedagogy, these insights are also relevant for thinking about teaching geometrical relations. Dynamic geometry software provides an immediate, accurate, exploration tool for making conjectures and verifying relations between elements of geometrical objects. As in Kullberg and Sun above, the dependency relationships we want students to learn about are available to be learnt if students are presented with varied examples of them so that conjectures can be generated and tested, and the behaviour and domains of the relationship experienced. So far we have seen different domains of communication: visual, symbolic, dynamic, gestural and verbal variation and how these can combine to give access to dependency relationships that underlie surface variation in examples.

How much variation, and variation of what?

I now move to an example of the use of variation in pedagogy in which considerations of the dimensions and ranges of variation became central. Koichu and his colleagues draw on the idea of a *space of learning* to consider the relationship between the intended and enacted objects of learning as an area for pedagogic decision. Their study is 'an application of variation theory to design of a task aimed at enhancing learners' awareness of mathematics as a connected field of study ... awareness of mathematics as a connected field of study was an intended object of learning' (Koichu, Zaslavsky and Dolev, 2013, p.461). Awareness here means the ability and disposition to discern, through and between different representations. They designed a sorting task with the twin aims of (a) completing the sorting and (b) undertaking analytical reasoning to do the sorting. There are several layers to this study: the task design as a tool for learning about similarities and differences; learners reflecting on the process to become aware of such similarities and differences in future; and the design process itself. Evidence from task outcomes, participants' comments in discussion, and records of their work gave insight into how they completed the task (53 subjects in three roughly equal cohorts). The design of the task was based on assumptions about the subjects' previous experience based on typical textbooks, and the need to control variations in sets of objects from which learners 'might observe regularities and differences, develop expectations, make comparisons, have surprises, test, adapt and confirm their conjectures within the exercise' (Watson & Mason, 2006, p.109).

Three types of controlled variation were introduced in the initial task design (Figure 5 shows 14 of the original 24 items to be sorted): mathematical objects to be sorted; representations including verbal descriptions and instructions; prior knowledge requirements to achieve the sorting. Although these variations appear to make pedagogical sense, and provide material for making important contrasts, at the design stage the critical aspects of the intended object of learning were not known.

|  |  |
| --- | --- |
| 9. Locus of points resulting from multiplication of y-coordinate of the points of the circle *x*2+*y*2=100 by 5/3. | 1. Locus of points such that the ratio of distances from the points to point (8,0) to the distances from the points to straight line x=12.5 equals 0.8. |
| 10. Locus of middles of the chords connecting the points on the parabola *y*2=40*x* with its vertex. | 2.  -3  -9 |
| 11. Locus of points such that the ratio of the distances from the points to point (6, 0) to the distances from the points to point (1, 0) is 1.5. | 3. Locus of points such that the ratio of the distances from the points to point (10, 0) to the distances from the points to straight line x=6.4 is 1.25. |
| 12. Locus of points such that the ratio between the distances from the points to point (3, 4) to the distances from the points to point (-3,4) equals 1. | 4. |
| 13. | 5. Locus of points such that the ratio of distances from the points to point (5, 0) to the distances from the points to straight line x=-5 equals 1. |
| 14.  10  6 | 6. Locus of points such that the ratio of distances from the points to point (0,0) to the distances between the points to straight line x=0 equals 1.25. |

Figure 5: part of Koichu et al.'s original sorting task (Koichu, Zaslavsky, & Dolev 2013)

With the first cohort, learners could achieve the desired sorting by focusing on algebraic manipulation and then classifying items by surface features instead of connecting different representations by using their underlying mathematical connections. The sorting had not been achieved by the analytical reasoning which would be evidence of the intended awareness. For the second cohort, items that had been treated algebraically were omitted and the cohort did not spend so much time on superficial similarities. However, surface features of the visual representations seemed to present obstacles in making sense of the verbal descriptions. From these observations it was decided that the third version should exclude the dimensions of variation that had hindered task completion, and focus on verbal descriptions of the main generating elements of each locus and other representations that had emerged as critical aspects in the earlier sortings (e.g. see items 4 and 13 in figure 5).

The relationship between this task and variation theory is complex. The intended object of learning is 'awareness' and although learners' awareness can be discerned phenomenographically (Marton & Booth, 1997) it cannot be directly varied in the presented task, since it is a property of learners and not of the mathematical items. 'Awareness of ... a connected field ...' might suggest that the connections themselves be varied, but because mathematics *is* a connected field it is hard to imagine how connectivity as a concept could be varied. Yet awareness is a legitimate objective for the teaching of mathematics, indeed Leung identifies a 'progression of awareness' in his work connecting Marton's four types of variation to dragging modes (Leung, 2008, p.153) finishing with fusion as the ultimate level of awareness. However, variation is a key component of Koichu et al.'s task design, since it has been used to generate the examples to be sorted, and their sorting is achieved through discerning similarities and differences in the set of examples. Furthermore, the designers reduced the set to one that contained only those features whose variation led to learners exhibiting the intended awareness of connectivity. It seems that variation in the parameters and conditions offered in the worded examples was enough to lead to the relevant awareness. Removal of distracting variations led to success. Koichu and his colleagues had hoped that learners would 'fuse' the understanding generated by algebraic forms, by graphs and by words during their sorting processes , but variation had to be successively limited towards variation *within* representations, rather than variation *of* representations, for the awareness of the embedded mathematics to be the lived experience of learning.

Mathematics pedagogy and variation theory

These reports have aired a range of issues that arise in the manifestation of variation theory in mathematics, and the use of variation in earlier texts. The decision to control deliberately the relationship between variation and invariance in a situation is pedagogic, and hence is in the realm of education rather than the mathematics itself, in which the variation relationships are inherent. The examples I have described above lead to some observations that extend variation theory in the context of mathematics education in ways that recognise the importance of dependency relations in mathematics:

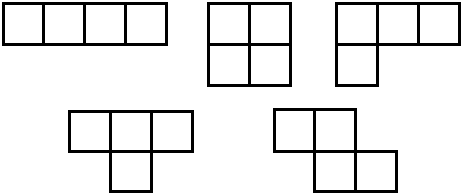
* the intended object of learning is often an abstract relationship that can only be experienced through examples; it is by observing the relationship between two varying aspects that the invariant relationship can be experienced and understood. By 'relationship' I do not mean the contextual fusion of two variables merely because they co-exist, but a *dependency relation* in which a change in one variable causes a change in another, when one variation necessitates another variation. Examples include the invariant relation between addition and subtraction (Sun above) or the action of dragging to change characteristics of a shape (Leung above) or when divisor being <1 or >1implies increase or decrease in a quantity (Kullberg above)
* characteristics of a relationship may vary in response to varied input, so the *behaviour* of the relationship varies for different inputs, but the relationship itself does not change (Kullberg)
* learners' action may not be the reflective, deliberate, action that is intended if it is easier to apply intuitive habits of mind (Koichu)
* when the intended object of learning is *awareness* identifying suitable dimensions of variation is difficult because awareness is a characteristic of the learner, not of the examples (Koichu)
* variation of appropriate dimensions can sometimes be directly visible, such as through geometry or through page layout (explicit), but often requires meaningful interpretation of symbolic forms (implicit). All the examples so far demonstrate aspects of this distinction. Koichu et al's work demonstrates the value of limiting the dimensions of variation, in their case to variations within representations to focus on meaning.
* the role of the teacher, or some other method, in drawing attention to connections, similarities and differences in the given examples introduces other dimensions of variation in the enacted object of learning (Kullberg, Sun, Leung, Koichu)

I hope I have shown above that if the object of learning is a dependency relationship, revealed by manipulation of variables, then higher levels of generalisation can be attained through pedagogic attention to variation and control of dimensions of variation. One possible criticism of the application of variation theory to mathematics pedagogy is that inductive reasoning cannot lead to a higher level of abstraction. Can students be scaffolded to work at a higher level of abstraction than can be provided by comparison and generalisation from given examples? By 'abstraction' I mean something beyond generalisation of mathematical relationships - I am not talking only of invariant background relationships here. 'Abstraction' in my use means that a dependency relationship itself becomes a kind of mathematical concept, so for example linear functions can be understood as dependency relations of a particular kind between two variables, or as objects in their own right with their own dimensions of variation, ranges of permissible change, operations, properties and so on (Watson & Mason 2004). In general these levels of mathematical concept are only available to us through speech and mathematical symbolisation. We cannot, for example, point to the concept of ratio; we can only point out situations in which ratio is the relationship between objects, and particular numerical instantiations of ratio, yet ratio as an idea has its own existence, definitions, variations and so on. Literature about this change of emphasis falls into two separate camps: the idea that cognitive change happens through processes of assimilation, accommodation and equilibration; or the idea that new objects can be brought into the communicable world through language and processes of enculturation. For me, variation theory provides part of an intellectual bridge to describe a combination of those processes, and also to inform design of learning environments. But the teaching of mathematics and presentation of near simultaneous examples that are constructed along the lines suggested in variation theory needs also appropriate forms of variance/invariant *language* and appropriate forms of variant/invariant *presentation*. We have already seen something of this in the examples so far, but now I shall demonstrate the importance of these considerations in achieving a shift to levels of higher abstraction.

Task sequence showing a trajectory through levels of abstraction

In the following sequence of tasks, dimensions of variation are controlled in such a way that a hierarchy of relationships is achieved, first through generalisation of dependency relations that have been exemplified, then through questioning which transforms the generalisation into a new object. In workshops with students, teachers at all levels, and teacher educators I have found these tasks to be almost universally effective in giving learners an experience of becoming more powerful with ideas that previously were abstract, within a context that is not dependent on advanced school mathematics curriculum knowledge.

The first task is to create what is called a 'tetramino' chosen from the connection of all possible tetraminoes. An analysis of these shapes leads people to the correct definition - four congruent squares joined at edges. In Marton's words learners are providing for themselves the 'necessary conditions of learning', that is they are aware that other numbers of squares are possible, and other ways of joining them.



Participants are then given a number grid on which to place their tetramino (Figure 6), and asked to identify the relationships between the four numbers that they have covered. So far, therefore, there is variation in shape and variation in the numbers covered. These fulfil two different purposes: the first is pedagogic, so that students are not using the same shape and can have a conversation about the different ways in which they have to express generalities later and how these relate to the shapes themselves; the second is based on the theory that we learn from examples, and that variation in those examples serves to draw our attention to what they have in common, by comparing them and/or reasoning inductively from them.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | **19** | **20** |
| **21** | **22** | **23** | **24** | **25** | **26** | **27** | **28** | **29** | **30** |
| **31** | **32** | **33** | **34** | **35** | **36** | **37** | **38** | **39** | **40** |
| **41** | **42** | **43** | **44** | **45** | **46** | **47** | **48** | **49** | **50** |
| **51** | **52** | **53** | **54** | **55** | **56** | **57** | **58** | **59** | **60** |
| **61** | **62** | **63** | **64** | **65** | **66** | **67** | **68** | **69** | **70** |
| **71** | **72** | **73** | **74** | **75** | **76** | **77** | **78** | **79** | **80** |
| **81** | **82** | **83** | **84** | **85** | **86** | **87** | **88** | **89** | **90** |
| **91** | **92** | **93** | **94** | **95** | **96** | **97** | **98** | **99** | **100** |

Figure 6: First grid and a chosen tetramino

They should then find that the *relationship* between the four numbers is the same, whatever position they have chosen on the grid for their tetramino, but different from the relations underneath other tetraminoes. This relationship can be expressed in general terms, such as: *n-1; n; n+10; n+11* using the tetramino shown next to the grid. The relationship depends on the shape and its position. However, the way we express the relationship can also vary. The same relationship can be expressed as *n; n + 1; n+11; n+12* by varying which cell is described as *n.* Students with the same shape can compare expressions; students with different shapes can compare relationships.

The grid content is then varied (Figure 7), but is mathematically similar in structure to the previous grid, i.e. consists of consecutive whole numbers. The task is repeated and the new relationship expressed, such as: *n-1; n; n+7; n+8.* Participants begin to realise that some of the numbers they have to use in the relationship are dependent on the grid size. The grid size has been varied, but the shape remains the same and the structure of the relationship stays the same. The object of learning at this point is the relationship structure, which stays the same even when the grid size varies. This use of variation confirms that we do not learn only about varied aspects, but about underlying relationships by becoming aware of similar structures in varied examples, as Sun describes (above)..

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **1** | **2** | **3** | **4** | **5** | **6** | **7** |
| **8** | **9** | **10** | **11** | **12** | **13** | **14** |
| **15** | **16** | **17** | **18** | **19** | **20** | **21** |
| **22** | **23** | **24** | **25** | **26** | **27** | **28** |
| **29** | **30** | **31** | **32** | **33** | **34** | **35** |
| **36** | **37** | **38** | **39** | **40** | **41** | **42** |
| **43** | **44** | **45** | **46** | **47** | **48** | **49** |

Figure 7: Varying the grid

At this point the teacher asks new forms of question which are about the relationship, such as:

On an 9-by-9 grid my tetramino covers 8 and 18. Guess my tetramino.

What tetramino, on what grid, would cover the numbers 25 and 32?

What tetramino, on what grid, could cover cells (m-1) and (m+7)?

These questions could be tackled through the materials, but are also accessible through abstract consideration of the relationship through looking at the structures of different generalisations. They are not questions which can be easily answered by repeating the original actions with different inputs unless a process of trial-and-adjustment is successful. New forms of reasoning are triggered which draw attention to the relationship, away from the initial manifestation; the particular relationship varies but its structure does not. This approach can be varied to other kinds of grid, such as bivariate grids, which I shall not consider further here.

For each successive task I deliberately vary a dimension of variation that would then become itself a parameter (a structuring feature) for a new task, creating a relationship between grid and shape that we could call the 'grid-shape' object. This new object, which is an abstract connection between grid and shape, can then be talked about, and new kinds of question posed about it, thus scaffolding abstraction. The task sequence illustrates: variation as a generator of examples for selection, comparison and generalisation; the use of the outcomes of generalisation as new objects which can themselves be varied; the twin roles of presenting variation and asking new questions that require deductive reasoning about new objects.

I have presented this task sequence as an example of how attention to variation can provide pedagogic strategies and pathways towards understanding mathematical ideas at higher levels of abstraction, a role for variation which is as yet under-researched. It has been hinted at in Marton's idea of fusion, but the outcome of fusion than needs to be seen as an object with its own behaviour and properties, and also in Leung's idea of level-2 invariance, but again the invariant idea has to become an object in itself.

CONCLUDING REMARKS

In this chapter I have brought together a wide range of mathematical tasks and reports that depend to some extent on implicit or explicit use of variation theory to generate learning and/or reveal the range of learners' understandings. In so doing I have included examples from textbooks to show that variation is an issue for textbook design as well as task design and pedagogy more broadly.

In all the examples I have given, there is more going on than merely asking learners to act on examples in which a critical aspect of an object of learning is varied. All examples address the harder problem of learning something about an underlying dependency relationship, which is very often the aim of mathematics teaching. This might be through: additional reflective tasks; contrasting behaviour in examples; pedagogic talk and gesture; juxtaposition of OPMS, OPMC and MPOS examples; direct connections between actions, visual layout and the relationship; limiting dimensions of variation; and avoiding variations that can be treated superficially. The variations themselves can contribute to inductive reasoning. On its own this form of reasoning is unlikely to lead to higher levels of abstraction, and might lead to unexpected inductive generalisations, but might also lead to useful conjectures about dependency relationships. I have also given a demonstration that higher levels of abstraction can be achieved through controlled variation, when the relationships identified in one cycle of variation become themselves the variable objects for the next cycle. In all these observations, the invariant qualities are as important as the variations, either as background, or as limiting factors, or as the relations that are often the aim of learning mathematics.

References

Al-Murani, T. (2006). Teachers’ awareness of dimensions of variation: A mathematics intervention project. In Novotna, J. (ed.) , *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 25-32). Prague: Charles University.

Cai, J. ,& Nie, B. (2007). Problem solving in Chinese mathematics education: Research and practice. ZDM-*International Journal on Mathematics Education, 39*, 459-473.

Godfrey, C. and Siddons, A.W. (1915). *Elementary Algebra Volume Ll*. Cambridge University Press, Cambridge.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: An effective way of mathematics teaching in China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp.309-348). Singapore: World Scientific.

Hewitt, D. (1998). Approaching arithmetic algebraically. *Mathematics Teaching, 163, 19-29.*

Huang, R., Mok, I., & Leung, F. (2006).Repetition or variation:Practising in the mathematics classrooms in China*.* In D. J. Clarke, C. Keitel & Y. Shimizu (eds.), *Mathematics classrooms in twelve countries: The insider’s**perspective* (pp. 263–274). Rotterdam: Sense Publishers.

Koichu, B. (2013). *Variation theory as a research tool for identifying learning in the design of tasks.* Retrieved 8 Aug 2014 from: http://www.mathunion.org/icmi/digital-library/icmi-study-conferences/icmi-study-22-conference/

Koichu, B., Zaslavsky, O., & Dolev, L. (2013). Effects of variations in task design using different representations of mathematical objects on learning: A case of a sorting task. In Margolinas, C. (ed.), *Task design in mathematics education, Proceedings of the ICMI study 22* (pp. 467-476). Retrieved 8 August 2014 from http://hal.archives-ouvertes.fr/hal-00834054 .

Kullberg, A., Runesson, U., & Måtensson, P. (2013). The same task? - different learning possibilities. In Margolinas, C. (ed.) *Task design in mathematics education, Proceedings of the ICMI study 22* (pp. 615-622). Retrieved 8 August 2014 from http://hal.archives-ouvertes.fr/hal-00834054

Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning*, 13, 135-157.

Leung, A. (2011). An epistemic model of task design in dynamic geometry environment. *ZDM - The International Journal on Mathematics Education*, 43, 325-336.

Leung, A. (2013). Thoughts on variation and mathematics task design. Retrieved 8 Aug 2014 from*: http://www.mathunion.org/fileadmin/ICMI/files/Digital\_Library/Studies/Thoughts\_on\_Variation\_and\_Mathematics\_Pedagogy.pdf*

Leung, A., & Lee, A. (2008). *Variational tasks in dynamic geometry environment* . Paper presented at the Topic Study Group 34: Research and development in task design and analysisICME 11. Monterrey, Mexico, July 2008.

Leung, A., Baccaglini-Frank, A. & Mariotti, M. A. (2013). Discernment in dynamic geometry environments. *Educational Studies in Mathematics, 84*, 439–460.

Margolinas, C. (ed.). *Task design in mathematics education, Proceedings of the ICMI study 22*. Retrieved 12th Aug 2014 from http://hal.archives-ouvertes.fr/hal-00834054 .

Marton, F. (2015). *Necessary conditions of learning.* London: Routledge.

Marton, F., & Booth, S. A. (1997). *Learning and awareness*. London: Psychology Press.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning Science*, *15*, 193–220.

Mason, J. (2000). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 97-111.

Mathematics Textbook Developer Group for Elementary School. (2005). *Mathematics*. Beijing: People’s Education Press.

Michener, E. R. (1978). Understanding understanding mathematics. *Cognitive science*, *2*(4), 361-383.

MME, Midlands Mathematics Experiment (1970). *CSE vol II part A,* London: Harrap.

Paterson, W. (1911). *School Algebra.* Oxford: Clarendon Press.

Sun, X. (2011). Variation problems and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Education Studies in Mathematics, 76*, 65-85.

Sun, X. (2013). *The fundamental idea of mathematical tasks design in China: the origin and development.* Retrieved 12th Aug 2014 from http://www.mathunion.org/fileadmin/ICMI/ files/ chinese\_variation\_theory-final-short-final.pdf

Watson, A., & Mason, J. (2004). The exercise as mathematical object: Dimensions of possible variation in practice. In *Proc. 24th Conf. of The British Society of Research in Learning Mathematics* (Vol. 2, pp. 107-112).

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91-111.

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