**What do we have to learn in order to learn mathematics?**

**Anne Watson, University of Oxford**

**Stirling 2009**

Many initiatives designed to improve learning in schools are generic and do not deal with subject-specific learning. I have been concerned for some time that unless we can describe ways of thinking which are specifically mathematical we raise test scores without improving deeper understanding and the capacity to learn higher mathematics. I start by looking at the characteristics of successful school mathematics learners, as identified by Krutetskii (1976). He was a psychologist working in the Soviet Union, and for 20 years tested students and watched them do mathematical tasks, working out what was special about those who did well at school. He found that these students were consistently able to:

* grasp formal structure
* think logically in spatial, numerical and symbolic relationships
* generalise rapidly and broadly
* curtail mental processes
* be flexible with mental processes
* appreciate clarity and rationality
* switch from direct to reverse trains of thought
* memorise mathematical objects (p. ??)

Interestingly, he did not find that calculation capability or memory for facts were necessary characteristics, although of course these speed up some aspects of mathematical work.

**Is ‘good’ teaching always good for mathematics?**

Currently in England good mathematics teaching includes several features which engage, include, and value all students. Typically, students’ knowledge and ideas are elicited through interaction with the teacher who then weaves these and builds on them to explain new ideas and tasks. Communication, interaction, discussion, expressing ideas and creativity are important in all lessons, not just mathematics, but I have found myself wondering whether these features guarantee that students will become more able to do the kind of thinking described by Krutetskii. Of course these do not automatically follow – it depends on what the teacher does. In particular, it matters what happens after students have expressed their own ideas. If the teaching focuses on the sharing and listening, there is no reason for individual ways of thinking to change.

Another feature of good teaching is the use of complex, exploratory, or real-life tasks to develop problem-solving skills. Many such tasks can be pursued using skills such as arithmetic, measuring, trial-and-adjustment and they finish when a suitable solution has been found, or a formula constructed. Again, as with sharing and listening, problem-solving does not *necessarily* lead students to use higher level tools, such as using algebra, comparing functions and so on – it is up to the teacher to lead students towards new methods by creating tasks and asking questions in which the power of new methods is necessary. For example, using a method to solve linear equations is only necessary if the answer cannot be guessed; using trigonometry to resolve a triangle is only necessary if we need a very accurate result or cannot measure.

The state of Western Australia developed a project called Working Mathematically in which extended tasks and contextual mathematics played a major role (Department of Education and Training, 2004; Stoyanova, 2007). Tests of large samples of students show varied success, and the tests included both straightforward tasks and those that need some interpretation and adaptation. The test scores were compared to reports of the frequency of strategies used by teachers. This revealed some interesting results.

Higher achievement was associated with:

* + asking ‘what if..?’ questions
	+ giving explanations
	+ *testing* conjectures
	+ checking answers for reasonableness
	+ splitting problems into subproblems

Achievement was not statistically associated with:

* + explicit teaching of problem-solving strategies
	+ *making* conjectures
	+ sharing ideas

Higher achievement was negatively associated with use of real life contexts, contrary to some other studies. This finding helped me realize that these differences were not to do with strategies *per se*. After all, the curriculum in the Netherlands is based in realistic contexts and their scores in international comparisons are fairly good. What seems to be important here is that the strategies are associated with higher achievement are those that necessarily involve changing the way students think. They push students beyond the obvious, to understand rather than perform mathematics, to see the effects of changing variables, to learn from applying results, and to restructure problems. Those that do not make an obvious difference are those that do not *force* new thinking, so are more dependent on the teacher pushing students to think more deeply about what they are doing and seeing – reflecting on the effects of their actions – to think in new ways.

**What new ways of thinking are necessary in mathematics?**

Consider this expression:

 49 + 35 – 35

Students’ responses to this kind of expression have been very well researched by a number of people (e.g. Jacobs et al 2007). Students have to be able to see such expressions as relations, rather than as instructions to calculate, in order to understand algebraic expressions. A central idea is the additive relation, and subtraction as the inverse of addition. Students who start calculating from left to right are at a disadvantage. In order to recognise that this is equivalent to 49, or even to 49 + *a* – *a* for any number *a*, students need to look at it as a whole, with an internal structure. In other words, they need to look at arithmetic algebraically. The problem with early algebra for many students is that they try to do the opposite – they try to look at algebra arithmetically.

These shifts of attention, from thinking about numbers to thinking about structures, and from calculating to identifying relations, may have to be made explicitly. With deliberate focus on relations, maybe more students will be able to, as Krutetskii suggested: grasp formal structure; think logically in numerical relationships; generalise rapidly and broadly; use shortcuts; use inverse trains of thought

Other changes are indicated in the following questions. To understand fully what follows it is a good idea to work these out and notice how you adapt various methods to the different number pairs[[1]](#footnote-1).

Find the numbers which midway between these pairs:

28 and 34

280 and 340

2.8 and 3.4

.00028 and .00034

1028 and 1034

38 and 44

-38 and -44

40 and 46

Typically, people find they use a range of methods: counting from both ends; visualising a number line; scaling; imagining moving along a numberline; working out how to adapt an earlier method rather than using a new one; flipping numbers across an imaginary zero, and so on. Very few add and divide by 2, even when they know this method well, because adding the numbers seems less elegant than adapting the original answer, and takes time. Several people report that it is enjoyable to adapt their insights to the different place values involved in some cases.

This sequence of questions highlights that students have to be able to change between physical and mental models; between images and symbols; from imagining a model to devising a rule; from devising a rule to using it as a tool in new contexts; from answering questions directly to looking for similarities. In terms of Krutetskii’s ideas, this sequence encourages learners to: grasp numerical structure; think logically in numerical relationships; shortcut processes; be flexible; switch from direct to reverse trains of thought.

My final example is the challenge to draw an equilateral triangle with its vertices in grid points on a 9-pin geoboard. Typically some students will draw this:

Often, students want to measure and, since measuring by its nature is never accurate, it is possible to imagine that the lines are equal. To *know* that this is not equilateral, even though in this orientation it may look like it, students have to shift from their initial visual responses to thinking about properties and relations between properties. This shift is identified by Van Hiele (1959) as essential in the growth of geometric understanding, and also in learning how to reason deductively rather than rely on sight. A teacher who helps students change from thinking ‘it looks like…’ to ‘it must be because …’ would be helping students learn to: think logically in spatial relationships; appreciate clarity and rationality; see objects as consisting of relations between properties.

It does not make sense to me to suggest that only the highest achieving students can think in these ways, and that they do it on their own and thus identify themselves as ‘gifted’. It makes much more sense, and did so to Krutetskii as well, to think that naming the ways of thinking that are characteristics of success make them teachable.

**Can students think in these ways?**

Fortunately, nearly all students learn in very natural ways which we can harness to make these shifts. Everyone within the normal range of brain function can:

* *Describe*
* *Draw on prior experience and repertoire*
* *Do informal inductive reasoning*
* Visualise
* Seek pattern
* Compare, classify
* Explore variation
* Do informal deductive reasoning
* Create objects with one or more particular features
* Give examples of what they know.

The first three of these ways of thinking do not lead to new ways of thinking, but the others, carefully harnessed by teachers, can enable students to think in new ways, or about new concepts.

**How do teachers help students adopt new ways of thinking?**

In our study of teaching in three schools that were trying to make a difference to their students’ attainment, we found that all teachers routinely made mathematical statements; elicit mathematical statements from students; ask learners to do things; direct their attention to certain ways of seeing and prioritising; and seek responses from learners (Watson, 2007; Watson and De Geest, 2008). Obviously the content of these actions makes a difference – it can be harder or easier, more or less searching. But there is more. Teachers also talk to students about the implications of the work that has been done, suggest connections, integrate the work done with other work, and do various things to affirm the outcomes of students’ efforts. The ways in which these are done varied widely. One teacher referred to a procedure as ‘a little bit of technology we need in order to think about a big question’. Another used a particular image in many different procedural contexts so that students would link them through visual imagery. We noticed that teachers who had been trained to teach mathematics were more likely to emphasise the mathematical importance, usefulness and context of what had been achieved whereas non-specialist teachers were more likely to talk about the necessity of certain techniques for tests, and to affirm the work done by praising effort and behaviour. This meant that some students would understand how their knowledge fitted with other mathematical ideas, and with what would come next, and the necessity for proof or other rigorous argument to back up empirical experience. It was often in the closing segments of the lesson that teachers embedded the idea that it was important to shift to more abstract and powerful ways of thinking, but some teachers had this as an agenda throughout their lessons.

The necessary ways of acting to learn mathematics have been called ‘habits of mind’ by Cuoco, Goldenberg and Mark (1996). They say that all mathematics learners need to develop the habits of:

* Pattern seeking
* Experimenting
* Describing
* Tinkering
* Inventing
* Visualising
* Conjecturing
* Guessing

They went further and identified some central algebraic habits, including:

* Liking algorithms
* Seeing calculations as structures of operations
* Representing classes of mathematical objects and their relations
* Extending meaning over new domains
* Abstraction

Some of these can be seen at work in your responses to the first two mathematical stimuli I offered above. Their characterisation of geometric thinking includes:

* Liking shapes
* Proportional reasoning
* Exploring systems and distinctions
* Worrying about change and invariance
* Reasoning about properties

Again, some of these had to be drawn on to think about the non-equilateral triangle.

In my experience, it is not enough to try to ‘teach’ these or merely design tasks which evoke them. More importantly, teaching needs to bring about *changes* in the ways students think - changes of habit you might say. For example, students need to be flexible between

* generalities and examples
* making change and thinking about mechanisms of change
* different points of view and representations
* induction and deduction
* using domains of meaning *and* using extreme values

I am in danger of providing too many lists of ‘things to do in lessons’ when what I really want to emphasise are the habitual ways of thinking that characterise mathematics. Hence they are habitual ways of thinking for mathematics teachers as well as for their students. If mathematics teachers keep on thinking like mathematicians, these awarenesses are bound to ooze throughout their teaching.

There are some other necessary shifts in ways of thinking, and I have not dealt with them all in detail here but provide them here for further thought. Good mathematics students have also to shift between:

* doing and undoing
* rules and tools
* representing and transforming
* discrete and continuous domains
* everyday and mathematical reasoning

Finally, rather than applying general ‘good’ teaching principles and general descriptions of ‘good’ thinking in mathematics lessons, to improve learning it is important to nurture those ways of thinking that are specially mathematical.

**References**

Cuoco, A. Goldenberg, P. & Mark, J. (1996). Habits of Mind: an Organizing Principle for Mathematics Curricula, *Journal of Mathematical Behavior* 15, 375-402.

Department of Education and Training, Western Australia (2004). *Student achievement in mathematics: Working Mathematically.* Retrieved Jan 19, 2008, from http://www.det.wa.edu.au/education/mse/pdfs/StudentAchievement%20WorkingMathematically.pdf.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L. and Battey, D. (2007). Developing children’s algebraic reasoning. *Journal for Research in Mathematics Education*, *38* (3),258-288.

Krutetskii, V. A. (1976) (trans. J. Teller). J. Kilpatrick & I. Wirszup (Eds.) *The psychology of mathematical abilities in school children.* Chicago: University of Chicago Press.

Stoyanova, E. (2007). Exploring the relationships between student achievement in Working Mathematically and the scope and nature of the classroom practices. In K. Milton, H. Reeves, & T. Spencer (Eds.) *Proceedings of the 21st biennial conference of Australian Association of Mathematics Teachers,* (pp. 232-240), Hobart, Tasmania.

Van Hiele, P. M. (1959). The child’s thought and geometry. In T. P. Carpenter, J. A. Dossey, & J. L. Koehler (Eds.) (2003) *Classics in mathematics education research,* pp. 60-66. Reston, VA: National Council of Teachers of Mathematics.

Watson, A. and DeGeest, E. (2008) Changes in Mathematics Teaching Project. Downloaded December 18th 2008 http://www.cmtp.co.uk/what\_were\_lessons\_like/

Watson, A. (2007) The nature of participation afforded by tasks, questions and prompts in mathematics classrooms. In L.Bills, J. Hodgen & H. Povey (Eds.) Research in Mathematics Education, vol. 9: Papers of the British Society for Research into Learning Mathematics***,*** pp. 111-126,London: BSRLM

1. I cannot recall where I first saw a sequence like this – it may have been in a presentation for the Association of Teachers of Mathematics. [↑](#footnote-ref-1)