Examining Teachers’ Use of (Non-Routine) Mathematical Tasks in Classrooms

from three Complementary Perspectives:

Teacher, Teacher Educator, Researcher

Ron Tzur Orit Zaslavsky Peter Sullivan

Purdue University, USA Technion, Israel Monash University, Australia

This Research Forum (RF) offers three complementary perspectives for examining how mathematics teachers use non-routine tasks in their classrooms. Following an introduction to the entire RF, Patricio Herbst presents a perspective of the teacher as a stakeholder in the symbolic economy of mathematics classrooms. Next, Peter Sullivan presents a perspective of a mathematics teacher educator through a model of task use and its implications for working with teachers. Then, Ron Tzur presents a perspective of a mathematics education researcher that focuses on how teachers’ epistemological stances impact their management of tasks. Finally, Anne Watson discusses aspects of the three perspectives and highlights additional considerations, including the benefit of engaging mathematics teachers in task design.

In the last three decades, along with the shift to reform-oriented approaches to teaching, a growing body of research has paid close attention to the design and implementation of tasks—problem situations, questioning methods, and activities for promoting student learning of mathematics (e.g., Ainley, Pratt, and Hansen, 2006; Henningsen and Stein, 1997; Hiebert and Wearne, 1993; Houssart, Roaf, and Watson, 2005; Simon and Tzur, 2004). This focus is a sound extension of the two dominant theories of learning, socio-cultural (Leont'ev, 2002; Lerman, 2006; Vygotsky, 1978) and constructivist (Confrey and Kazak, 2006; Piaget, 1985; von Glasersfeld, 1995), as both contend that learners’ goal-directed activity is the source for conceptual advance. That is, in reform-oriented approaches, tasks play the key role of interface between teacher intentions and student activities and attainments. Such an interface is needed because, as Pirie and Kieren (1992) contended, a teacher can occasion students’ learning only indirectly, through engaging them in situations that prompt non-linear progressions toward intended mathematical understandings.

Kilpatrick et al. (2001) maintained that the quality of teaching depends "on whether teachers select cognitively demanding tasks, plan the lesson by elaborating the mathematics that the students are to learn through those tasks, and allocate sufficient time for the students to engage in and spend time on the tasks" (p. 9). This, in a nutshell, highlights the challenges teachers face in their choices, design, and implementations of instructional tasks. Although there are teachers who generate tasks on their own, most interpret and implement tasks generated by others (math educators, curriculum designers, etc.). Consequently, there are often discrepancies between designers’ intentions and the actual implementation. Our Research Forum offers novel ways and considerations for examining, from three complementary perspectives, how and why teachers interpret, alter, and use non-routine, inquiry-promoting mathematical tasks in their classroom,:

1. A mathematics teacher's perspective that considers constraints within which teachers operate, their goals and beliefs, and their degree of confidence and flexibility (Herbst);

2. A mathematics teacher educator's perspective that considers the task as a way for conveying desirable teaching goals, promoting students' learning, and providing mathematics teachers with feedback and guidance that may help them transform their teaching (Sullivan);

3. A mathematics education researcher's perspective that analyzes characteristics of tasks and ways in which tasks unfold in the classroom, particularly focusing on epistemological assumptions that underlie teachers’ use and alteration of tasks (Tzur);

At times, these perspectives may seem inseparable, just as mathematics educators may hold several roles, i.e., teacher, teacher-educator, and/or researcher. Yet, each of the presenters in the following contributions highlights a particular perspective. The discussion and synthesis of key issues that emerge from all three perspectives (offered by Watson) provides insight into different aspects of task design and implementation, and adds to the bridging between theory and practice as well as to the identification of aspects that require additional scholarly attention (e.g., teachers' sequencing of tasks). Consequently, the significance of this RF lies in the coordination among different perspectives, and the broader and more complex picture they present in terms of understanding discrepancies between intended and implemented classroom activities and norms.

In the group discussions that will follow each presentation, we will address the following questions, as well as other questions that the audience will raise:

a) In what ways are (non-routine) tasks that teachers use in their classroom similar to or different from the intended tasks suggested by mathematics teacher educators and curriculum developers? What kinds of discrepancies between the intended and the implemented tasks can be identified? What is lost/gained by teachers' modifications of tasks?

b) What explanations can we offer to account for such discrepancies? Can regularities in characteristics of tasks that teacher alter be identified/explained?

c) How might teacher modification of tasks serve in inferring into and promoting their pedagogies? What can mathematics teacher educators and researchers offer teachers to support and enhance their engagement in task adaptation that promotes student learning?

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the teacher and the task

Patricio Herbst

University of Michigan, USA

Why would a teacher make changes to a task? What is at stake for a teacher in a task? I propose to consider the work of teaching as one of effecting academic transactions in which the teacher is accountable to both the mathematical meanings represented in a task and the opportunity to learn that the task offers to students.

I examine the role of tasks from the position of the teacher, eventually coming around to the question of why a teacher might adapt or change a task designed by developers or researchers on learning. I take my charge as a researcher of teaching who endeavors to understand the rationality of teachers, making it clear that inasmuch as this rationality is often unspoken and tacit (see Herbst and Chazan, 2003), I am producing a theoretical model rather than relaying a testimony. In my own work (Herbst, 2006), I make a distinction between two frequent uses of the word task referring to one as *problem* and to the other as *task*. I use *problem* to refer to the mathematical statement of the work to do. For example, I consider “given two intersecting lines and one point on each of them (but not on the intersection), draw a circle tangent to both lines, so that the given points are its points of tangency” as a problem. I use *task* to refer to the (anticipated or observed) deployment of one such problem over time, in the actions and interactions of particular people (say in a US high school geometry class), doing particular operations with particular resources.

The first notion (‘problem’) echoes Brousseau’s (1997, p. 79) epistemological notion of problem as a counterpart of a specific mathematical idea. In the example, the problem is the counterpart of a theorem (the “tangent segments theorem”) that specifies the necessary and sufficient conditions on which such construction is possible: a circle exists which is tangent to two intersecting lines at two given points if and only if the two given points are equidistant from the intersection. The second notion (‘task’) builds on Doyle’s (1983) proposal to study the curriculum by describing the work done in classrooms. From Doyle we get the observation that the work done (and thus the opportunity to learn) may be different depending on the overt goal proposed (e.g., to produce a circle tangent), the resources available, and the viable operations (all of which echo Brousseau’s notion of the *milieu*). Doyle also noted that a task plays a role in the accountability system in the class, where accountability refers to how much value a task had for students (i.e., in terms of grades). I explore accountability from the perspective of a teacher. What is at stake for a teacher in a task?

The notion of task as the deployment of work on a problem over time and in an institutional space is a common ground for each of the three vertices of the instructional triangle: mathematics, the students, and the teacher (see Cohen, Raudenbush, and Ball, 2003). Mathematically, insofar as tasks are segments of social practice, one can see tasks as embodiments or representations of mathematical ideas, much in the same way that performances in art or dance embody ideas (see Herbst and Balacheff, in press; Lakatos, 1976). As far as the students are concerned, tasks are not performances to contemplate but opportunities for students to become, to come to know more or differently. As the introduction to the Research Forum argued, after Pirie and Kieren (1992), “a teacher can occasion students’ learning only indirectly, through engaging them in situations that prompt non-linear progressions toward intended mathematical understandings.” That is, tasks are opportunities for students to act and possibly learn.

If one accepts those as descriptions of how mathematics and the student have a stake in a task, it also follows that there is a kind of tension between the two conceptualizations. One could imagine, for example, a scripted classroom discussion where students and teacher elegantly acted out the emergence of a solution to a problem. And one could contrast that image with another classroom, where long silences extend while students struggle with that same problem, the teacher resists giving away hints, some students solve a different problem while others give up, etc. While the first scenario might illustrate how *the work on the problem* (the task as performance) embodies a piece of mathematical knowledge, the second one illustrates what the room could look like when students are given the opportunity to progress nonlinearly “toward intended mathematical understandings,” which surely has to include at least as a possibility that such nonlinearity might take them to unintended places. None of the two scenarios is realistic or desirable, but they help make the case for a teacher who acts rather than withdraws (Smith, 1996), and introduce their role and stake vis-à-vis the task.

What is at stake for a teacher

A teacher is responsible to manage the tension that a task presents in those two senses. She is responsible for the task as a representation of the *mathematics* to be learned and for the task as an opportunity to *study* and *learn* that mathematics. I conceive of classrooms as symbolic economies: Classrooms are places where transactions take place between the work that people do and the mathematics that they lay claim on. The teacher manages this economy—she manages transactions between work done and knowledge acquired. The teacher also has a stake in a task.

When teaching mathematics in school, a teacher is bound to mathematics and to students by a didactical contract. Any didactical contract gives the teacher a privileged position in organizing the work that the class will do over the duration of the course of studies. Thus, the teacher is *entitled* to decide what will be done, when, and for how long; and, for the same reason, she is also *accountable* for that work. From the teacher’s perspective, a task is a bid to fulfill some of his or her contracted responsibilities. In engaging her class in a task a teacher exercises her entitlements and also submits to the responsibilities that those entitlements carry. It is not a risk-free venture for a teacher, at the very least because it entails an investment of time, a scarce, non-renewable resource in the duration of a course of studies within a school.

The work of developers creating mathematical tasks, and of researchers who focus on student learning through tasks can serve as resources for a teacher. Such work helps argue for the goodness of investing class time on some tasks. But they don’t relieve the teacher from the responsibility to account for the time spent on one such task and to manage the process by which such task will deliver what it has promised. Moreover, what is at stake is not only time (invested, wasted, or unused). The learning opportunity to be experienced by students in that time and the mathematics to be produced with students in that experience are at stake as well: They are not automatic earnings derived from the decision to engage in a task, they could be earned, shortchanged, or even lost depending on what happens in action. The choice to spend a certain amount of time working on a problem might be a defensible investment initially. But management has to be active during its deployment to make the investment work out. And, among other things, active management might recommend second-guessing that investment, suggesting that new things must be done in order to sustain the soundness of the investment. The point is that a teacher who honors her or his professional responsibility in the didactical contract is accountable for attending to whether and how a task fulfils its promise as it develops over time.

EXPLAINING CHANGES IN TASKS

Those problems of accountability and management are proposed here as an explanation for why practitioners may change the task in ways that puzzle researchers on learning or curriculum developers. Accountability and management are not necessarily conscious problems for a practitioner, so one might not be able to elicit them as declarations of belief or goals. The specifics of how they are handled are likely dependent on individual teacher knowledge and beliefs, but their existence as problems is a characteristic derived from the institutional position of the teacher and the rationality of practice. The problems may not apply equally to teaching outside of schooling. The problems are proposed as elements of a theoretical model of the role of the teacher, but they can be confirmed empirically.

Let me illustrate this argument with a concrete example. The example is a set of possible classroom episodes that include a task that could unfold as a class works on the tangent circle problem (see above). As part of our study of practical rationality of mathematics teaching (Herbst and Chazan, 2006; Herbst and Miyakawa, in press; grip.umich.edu), we have produced an animated story of cartoon characters (The Tangent Circle) and comic book variations of that story to represent possible ways in which that task could unfold. In the 11-minute animation, a teacher reminds students that on the previous day they had learned that the tangent to a circle is perpendicular to the radius at the point of tangency. The teacher asks them to draw a circle tangent to two given lines at two given points that appear not to be equidistant from the point of intersection. Some students draw a circle without a compass, forcing it to be tangent at the expense of making it look unlike a circle (see Figure 1a), whereas other students draw circles with a compass at the expense of not achieving any one of them that looks tangent (see Figure 1b).

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| Figure 1a. Alpha’s circle | Figure 1b. Rho’s circle |

One student (Lambda) claims early on that it is impossible to solve the problem and suggests moving the points to be able to do it; other students scorn her for changing the problem and the teacher lets her claim of impossibility fade out. The teacher poses another problem to the whole class: Given two intersecting lines and a point on one of those lines, where should we plot the other point in order to construct the tangent circle? A 5-minute dialogue ensues that over time elicits viable and unviable ideas from students and implements those in constructions until an idea appears to choose points equidistant from the point of intersection and draw perpendiculars to find the center. The teacher then says, “what we just did is, we discovered a theorem” and writes on the board “if two intersecting lines are tangent to a circle, the points of tangency are equidistant from the point of intersection.” We have created alternative representations of this story. Among these we have varied the initial conditions of the problem (giving no points of tangency, giving 1 point, or giving 2 points that appear to be equidistant) and we have also created a “short version” where two non equidistant points are given but the teacher moves to state the tangent segments theorem immediately after Lambda says that the problem is impossible. We use these representations of teaching as prompts for experienced teachers to comment on the decisions made by the cartoon teacher. Analysis of that commentary is ongoing, to document whether and how teachers perceive these problems of accountability and management. In what follows I illustrate how the media can prompt teacher commentary that confirms the existence of those problems.

The problem of “accountability” is at play in the decision whether and why to make time, between the installation of the radius-perpendicular-to-tangent theorem and the statement of the “tangent segments theorem,” to work on a variation of the tangent circle problem. If so, this period will engage students in doing something that they might or might not be able to do (depending on the choice of the givens); yet the success or failure doing the construction does not correlate with the success or failure of getting the new theorem on the table. What will that time count for? The teacher has to justify this investment a priori as well as to monitor its efficiency leading to the intended theorem. Epistemological and learning arguments might (and often times do) persuade a teacher that time would be well spent in this task: These arguments could draw, for example, on the value of creating in students a sense of intellectual need for the new theorem (see Harel and Sowder, 2005). The difference between the animation and the “short version” highlights how this problem is an open problem for the teacher. From the developer’s or the learning theorist’s perspectives it may be clear that more could be done after Lambda says that one could construct the circle if one moved the points. But for the teacher the issue is whether what has already happened suggests that it is time to cash the investment made. In other scenarios the need to account for the investment of time might press the teacher to write off a loss (of time) before it is too late. The problem is grounded in the assumption that sooner or later the teacher will be accountable for the time spent and the hypothesis that a teacher experiences larger investments of time as deserving larger “cash” value. The “cash” value alluded consists of claims that the teacher can lay on the class’s knowledge of mathematics. This takes me back to the problem of management which requires the teacher to manage two senses of task noted above—the task as representation of mathematics and the task as opportunity for students to learn.

The problem of “management” is really a multitude of problems and it refers, to be quite clear, to the management of knowledge and learning, rather than just to the management of behavior. A teacher is an observer of the activity that exists in the classroom and can therefore attest to its mathematical value. But a teacher is also an actor in sustaining that activity with the students and can attest to its cost. The teacher needs to manage tensions that arise from that double identity (Herbst, 2003; also Ball, 1993). In the story described above it may be apparent to the reader that when the teacher states the theorem that they “discovered,” that is not quite an appropriate assessment of what they actually did. Even if one ignores for a moment that they only verified perceptually that the circle is actually tangent, the construction really asserts the possibility to find a tangent circle as long as the points of tangency are equidistant from the point of intersection. In contrast, the theorem stated assumes the tangency and claims that the points are equidistant. A more accurate reading of what they actually achieved is that they have a better action model of what is good to have in order to do the construction. It would probably take more time and more tasks (an adidactical situation of formulation and later an adidactical situation of validation, in the sense of Brousseau’s, 1997, p. 65) to claim that the aggregate work actually has the mathematical value of “discovery of the tangents theorem” (as a statement validated by a mathematical theory).

But to hold off effecting that transaction (i.e., to expect more work before claiming that the theorem has been discovered) might incur in extra costs as regards to the nature of the opportunity to learn that the teacher needs to sustain. These costs might include, for example, exacerbating the individual differences among students in regard to what they understand the goal of the task to be, what they think the resources or the operations needed are. All of these differences are part and parcel of what a learning theorist looks after, to understand and to document. But for the teacher of a class, who is responsible to teach the same curriculum to all students, these differences presage management nightmares. In the story one can see a glimpse of that by comparing how much earlier than her peers Lambda came to the realization that one had to move the points: The teacher makes the choice of giving another task rather than “cashing” the theorem on account of Lambda’s comment, but the learning cost that the teacher has to manage in that case is one of sustaining attention to at least two interpretations of the goal of the task (to draw a tangent circle in the given conditions, to find the conditions on which one can draw a tangent circle) that span the learning environment.

UNDERSTANDING THE work OF TEACHING

The curriculum developer and the researcher on learning are task stakeholders just as the teacher is. They may be frustrated with the changes that the teacher makes on the moment. They may think that those changes come from ill will or poor knowledge. While some times those reasons may aggravate matters, I hope I have made the case that if changes do occur, those can be explained by understanding better the work of teaching. The teacher is a stakeholder of the task in that she needs the task to be instrumental to the institutional goals of teaching, which quite often mean communicating some specific mathematical ideas to all of a diverse group of children within a particular space and a set amount of time. These goals are related but not reducible to the goals of representing mathematics or occasioning individual learning. The teacher needs to use the task to fulfil those goals. The need to handle the problems of accountability and management may explain why at times such use of the task may run against the expectations of developers or learning theorists and why some other times a teacher may just choose not even to try. Researchers and developers could be more deferential, accepting that the teacher is really busy solving her own work problems. Both curriculum development and the study of learning in classroom settings need to be better informed by descriptive (rather than normative) theories of teaching.

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Designing Task-based Mathematics Lessons as Teacher Learning

Peter Sullivan

Monash University, Australia

One of the challenges in all educator-led mathematics teacher learning is to create awareness of opportunities afforded by non-routine tasks, while at the same time fostering appreciation of the constraints in implementing such tasks in classrooms. The intention is that when teachers have opportunities to use non-routine tasks and associated pedagogies in classrooms, they will be aware of the advantages and potential of such tasks as well as anticipating potential barriers to implementation.

Introduction

I am assuming that: both prospective and practising teachers are interested in talking about tasks and lessons; it is a substantial and under-recognised challenge to convert potentially engaging tasks into productive learning through particular teacher actions; and, by studying the processes of constructing lessons, teachers can come to see the potential in non-routine tasks, and also the constraints that they might experience in using them.

This process of converting documented tasks to student learning was described by Stein, Grover, and Henningsen (1996), who analysed 144 tasks in terms of their features and cognitive demands, and studied the implementation of the tasks in classrooms. Their process has the task—influenced by the teacher goals, subject matter knowledge, and knowledge of students—informing the lesson (meaning the task in the classroom) which, influenced by classroom norms, task conditions, teacher instructional habits and dispositions, and students’ learning habits and dispositions, creates the potential for student learning. The process they outline can assist prospective and practising teachers to appreciate the importance of theories of learning mathematics, and the ways that theories can inform classroom practice. Of course, the implications of theories for practice need to be made explicit, and one approach to this can be to link tasks and pedagogies to classroom practice. Essentially the intention is to offer practically experimented exemplars of non-routine tasks, and to study the implementation of those tasks in classroom, even over some iterations.

The following draws on results from a project researching the implementation of a particular type of non-routine tasks in classrooms, and a resulting recommended model for planning and teaching mathematics. This model can form the basis of collaborative approaches such as learning study (Runesson, in press), study groups (Arbaugh, 2003), coaching (Fullan, 2000), and Japanese lesson study (e.g., Stigler, and Stephenson, 1994). The advantage of incorporating the planning and teaching model into those formats being that there are aspects of using non-routine tasks in classrooms that are far from obvious. For example, unless teachers are aware of processes such as managing post-investigation discussions to facilitate making connections and forming generalisations, or adapting tasks to support learners experiencing difficulties, or ways of building a culture of collaboration, then key affordances of using non-routine tasks may be missed.

The *OVERCOMING BARRIERS* project

The model that can be used as the basis of structured teacher learning was an outcome of research that identified and described particular aspects of classroom teaching that may act as barriers to mathematics learning for some students. The project first drew on responses from focus groups of teachers and academics to suggest strategies for overcoming such barriers (see Sullivan, Zevenbergen, and Mousley, 2002). Next, the project analysed some partially scripted experiences taught by participating teachers (see Sullivan, Mousley, and Zevenbergen, 2004). This analysis allowed reconsideration of the emphasis and priority of the respective teaching elements. Eventually the project researched ways teachers adapted the model to their classrooms.

There are five key elements of the model: the tasks and their sequence; *enabling* prompts that offer a particular approach to supporting students experiencing difficulty; *extending* prompts that can be used to challenge students who have completed the set work; making implicit pedagogies explicit; and the building of mathematical community.

The tasks and their sequence

Many commentators (e.g., Christiansen and Walther, 1986; Brousseau, 1997) have argued that the choice of tasks is a key element of any planning. As implied by Vygotsky’s (1978) zone of proximal development, one aspect of the teacher’s task is to pose to the class problems that most students are not able to do. While this model is applicable to any non-routine tasks, the project was based on a particular type of open-ended task, which can be illustrated by an example:

After 5 games, the mean number of points that a basketballer had shot was 6, and the median number of points was 4. What scores might the basketballer have shot in each game?

This task is non-routine in that it is not readily solved by the application of a formula, and students must consider the meaning of the concepts of mean and median. Assuming that students have met mean and median, it has an easy entry in that students can choose possible scores with which they are familiar, and there are obvious and ready extensions possible for students who find a few responses quickly. Such tasks are content-specific in that they address the type of mathematical operations that form the basis of textbooks and the conventional mathematics curriculum.

Connected to the choice of task is what Simon (1995) described as a hypothetical learning trajectory made up of three components: the *learning goal* that determines the desired direction of teaching and learning, the *activities* to be undertaken by the teacher and students, and a *hypothetical cognitive process*, “a prediction of how the students’ thinking and understanding will evolve in the context of the learning activities” (p. 136). In the case of the example task, this might involve selected preliminary experiences such as, for example, posing tasks exploring mean and median separately, and illustrating trial-and-error processes, and also planning what might come after this task such as transfer to different contexts, practice to fluency, introducing the concept of mode, and even box plots.

Enabling prompts to support students experiencing difficulty

If the teacher chooses tasks that most students are not able to do, as is desirable, there is a need to consider processes for supporting students who may not be able to complete the tasks even with adult guidance. The model suggests that teachers offer *enabling prompts* to allow those experiencing difficulty to engage in active experiences related to the initial goal task. Enabling prompts can involve slightly lowering an aspect of the task demand, such as the form of representation, the size of the numbers, or the number of steps, so that a student experiencing difficulty can proceed at that new level, and then if successful can return to the original task. This approach can be contrasted with the more common requirement that such students (a) listen to additional explanations; or (b) pursue goals substantially different (less demanding) from the rest of the class. In the project, the use of enabling prompts generally resulted in students experiencing difficulties being able to start (or restart) work at their own level of understanding and allowed them to overcome barriers met at specific stages of the lessons. As an example of an enabling prompt for the task above, the teacher might invite a student to work out what might be scores if there are 5 games and the mean number of points shot is 6. The effect of this enabling prompt is to reduce the variables from two to one, while preserving the open-ended nature of the exploration. The teacher might also say that the mean is 6 and the median is 4, without specifying the number of games. This removes one of the constraints, and so the task is one step easier.

Extending prompts for students who complete the initial task readily

If the task is at the appropriate level of challenge for most students, there may well be students who complete the task quickly. Teachers can pose prompts that extend students’ thinking on the initial task in ways that do not make them feel that they are getting more of the same or being punished for completing the earlier work. Students who complete the planned tasks quickly are posed supplementary tasks or questions that extend their thinking. An example of an extending prompt for the above task could be:

What if I told you that the mode was 7 as well?

The effect of this prompt is to examine the impact of the additional constraints. Another example, which introduces the case of median when there is an even number of scores, could be:

What if there had been 6 games?

Explicit pedagogies

Especially with non-routine tasks, the model assumes that it is critical that teachers make explicit for all students the usual practices, organisational routines, and modes of communication that impact on approaches to learning. These include ways of working and reasons for these, types of responses valued, views about legitimacy of knowledge produced, and responsibilities of individual learners. As Bernstein (1996) noted, through different methods of teaching and different backgrounds of experience, groups of students receive different messages about the overt and the hidden curriculum of schools. Sullivan et al. (2002) listed a range of particular strategies that teachers can use to make implicit pedagogies more explicit and so address aspects of possible disadvantage of particular groups. An example in the case of this type of task is for the teacher to explain to students that not only are multiple solutions possible, but they are desirable. Likewise, for example, students can be invited to be creative, to consider the appropriateness of trial and error methods, and to discuss the role of basketball in the question.

Learning community

A deliberate intention in the model is that all students progress through learning experiences in ways that allow them to feel part of the class community and contribute to it, including being able to participate in reviews and class discussions about the work. It is assumed that all students will benefit from participation in at least some core activities that can form the basis of common discussions and shared experience, both social and mathematical. It was also clear from the research that teachers can take particular actions that can support or inhibit the building of community. Teachers, in observed reviews of student work, for example, would often invite students to contribute randomly and so would not be aware of the nature of the contribution that the particular students would make. Further, it was common for teachers to fail to interrogate students, or encourage other students to do so. The net result was that there was little sense of a learning community developed. In the case of the example above, it is assumed that teachers would want to ensure that a student who found an answer by random trial-and-error would be invited to describe their responses, and perhaps another student who had systematically determined a range of responses, and another who had sought a generalised response could also be called on.

Use in teacher education

The *Overcoming Barriers* project demonstrated that both primary and secondary teachers are able to implement the planning and teaching model in everyday classrooms. The model can be adapted to the methods of studying tasks and lessons that are used as the basis of many teacher education programs. Some of the key elements could include teachers:

- studying the nature of tasks, and especially ways in which non-routine tasks are different mathematically and pedagogically from conventional tasks;

- considering the affordances and constraints in using non-routine tasks;

- demonstrating the planning and teaching model through a mathematics “lesson” delivered to the participants at teacher learning sessions;

- collaborative planning of other hypothetical lessons, with no intention that they be taught, with critical review of those plans;

- forming small groups to plan and then teach a lesson, incorporating iterative processes for review; and

- creating opportunities for review and reflection not only on the teaching and planning model but also on the teacher learning process itself.

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A Researcher Perplexity: Why Do Mathematical Tasks Undergo Metamorphosis in Teacher Hands?

Ron Tzur

Purdue University, USA

The central argument of this essay is that, in order to understand different ways in which mathematics educators (MEs) and mathematics teachers (MTs) interpret and use instructional (non-routine) tasks to promote student learning, it is necessary to account for their epistemological stances. I identify three ways by which MTs alter tasks and propose three plausible sources for such alteration. The third source, MTs’ epistemological assumptions and considerations, is examined through their conceptions of the goals of teaching and the activities used to accomplish those goals. Using the distinction between perception- and conception-based perspectives as a lens, I address the different use of tasks for linking students’ extant conceptions with those to be learned, and propose three key implications of such differences.

As a researcher interested in better understanding how mathematics teachers develop learner-empowering pedagogy I am perplexed by the phenomenon that is the focal-point of this Research Forum. Why do teachers enact mathematical tasks designed by mathematics teacher educators (METs) in ways that substantially deviate from how those MTEs (a) would enact the tasks themselves or (b) anticipate the tasks to unfold in a teacher’s classroom? Addressing this problem is important because of the key role assigned to mathematical tasks—the tool by which teachers can nurture student learning (Sullivan et al., 2004; Zaslavsky, 2007). MTEs’ scholarship repeatedly demonstrates how non-routine tasks, when enacted adeptly, succeed in fostering the desired quality of mathematical understandings whereas, for too many students, current practices fail. When the alternative tasks undergo metamorphosis the quality of student mathematics may be harshly compromised, hence the necessity to examine the issue.

Like Watson (this forum), the premise of my argument is that a teacher is always responsible for tailoring tasks to the unfolding of students’ work in a mathematics classroom. Likewise, I assume that a MT alters tasks in service of genuine, best intentions to foster student learning while coping with constraints she or he faces (Herbst, this forum). Thus, I turn to articulating differences inherent to MTs’ and MTEs’ teaching, that is, assumptions and considerations that underlie their use of tasks. This includes three common ways in which teachers seem to alter a task and three plausible sources for these alterations, theoretical constructs for analyzing the third source (epistemological stance) that in my judgment is most resistant-to-change, and implications of this analysis.

Ways and Sources of Task Alteration

My work with MTs pointed to three characteristic ways in which they modify tasks. First, when planning, teachers often conclude that a task, as designed, is not likely to accomplish their goal for student learning and they adjust it before the lesson to fit with their own anticipation of how student learning unfolds. Second, a teacher may begin a lesson with a task enacted as intended, but, sooner or later, interpret students’ work on the task to indicate lack of progress, which leads to task renegotiation. Third, teachers may plan and enact a task believing that their teaching corresponds to how they saw it used by a ME and/or to the designer’s expectations, whereas a MTE who observes the lesson cannot but notice striking differences. Regarding all three, Stein et al. (1996) found, for example, teacher tendencies to reduce the demand/challenge level of a task. As much as those three types of task alteration differ, an underlying feature common to all three is the relationship between the teachers’ goal and the regulation of their actions with the task as a tool to accomplish their goal. Let me illustrate the point with an observation from Dan’s lesson.

Dan was a mid-career, enthusiastic, grade-6 teacher whose lesson on division of whole numbers I studied. He was not math anxious, deeply appreciated its importance and beauty, adhered to reform-oriented methods, enjoyed solving challenging problems, and hoped for his students to develop similar dispositions. While planning, he thoroughly revisited the mathematical concepts he himself learned the previous summer in a reform-oriented workshop. He clarified for himself the big idea (Schifter et al., 1999) of division as a crucial component of multiplicative reasoning, and worked out examples of long-division algorithm with Base Ten Blocks. In the previous lesson (Friday), the class solved partitive division problems using the blocks. For Monday, then, he planned to recap Friday’s lesson, have three students solve a long-division problem on the board, then engage the class in understanding each step in the algorithm. The first two parts took about 20 minutes (as planned). However, as Dan turned to the third part, which he anticipated to be a straightforward work on linking two known processes, he realized students were ‘lost.’ Several times he mentioned what they did on Friday, growing frustrated with their inability to connect it with the algorithm, then asked them to bring out and use the blocks for solving the problem. They did, but could not yet ‘see’ the link to the algorithm that was painfully obvious to him. Extremely frustrated, Dan shifted to the traditional method (show-and-tell) that *he despised*, directly pointing out step-by-step correspondences between the activity with the blocks and the algorithm.

An example of the second type of task alteration, Dan’s lesson sheds light on three plausible sources for this phenomenon. A teacher’s prime goal is to foster student learning of particular mathematical ideas within an environment that presents many real and/or perceived constraints (Sullivan, this forum). The goal a teacher sets and the activities she or he takes to accomplish it depends on her or his assimilatory conceptions, namely, understanding of mathematics and pedagogy when planning, implementing, and adjusting situations (Ball, 2000; Shulman, 1987; Tsamir, 2005), as well as institutional norms and practices (see Herbst’s excellent discussion of some critical constraints, this forum). Consequently, three plausible sources may effect task management: (a) the teacher’s conceptions of the mathematical learning goal that a task is designed to promote, (b) her or his facility with using the task as a pedagogical tool, and (c) the teacher’s implicit or explicit epistemological stance as to how a person comes to understand a mathematical idea she or he does not yet know and the role a task plays in this process.

Dan’s example is telling because it demonstrates the logical status of the first two sources—they are necessary but insufficient. Dan’s very strong mathematical understanding was clearly on par with what the MTE community yearns. He was also highly competent in using reform-minded problem solving processes and questioning techniques, small group and whole class discussions, and technological tools. Yet, his reasoning about the planned tasks as well as his reflections on the growing dissatisfaction from the impact of task modification on students’ progress indicated epistemological commitments that markedly differed from the MTE’s stance. The work with numerous teachers like Dan convinced my colleagues and me of the need to seriously examine this third source.

Task management and Teacher Epistemological Stances

Because a mathematical task is a strategic means for accomplishing the teacher’s goal of student learning, a teacher’s understanding and managing of a task (in the sense articulated by Herbst, this forum) depends on her or his idea of learning, that is, one’s implicit or explicit epistemological stance (Tzur, in press). Apart from the traditional show-and-tell perspective, a research team of which I was part (see Simon et al., 2000; Tzur et al., 2001) postulated that practices and thinking of teachers like Dan who attempt to adopt reform-oriented pedagogies could be rooted in *perception-based* or *conception-based* perspectives. A perception-based perspective is marked by a noticeable transformation in a teacher’s traditional practice due to adopting a view of learning as an active process (e.g., heavy use of manipulatives). This change, however, is not accompanied by a change in the teacher’s view of the epistemological status of mathematical knowledge and what in students’ activities enable its formation. Like in the traditional perspective, the underlying premise of a perception-based perspective is that the mathematics a teacher came to know/understand has existence of its own independent of the person who knows and how she or he came to know it.

Such a view makes sense if one considers, for example, how Dan formed his understanding of the long division algorithm, including his excitement for gaining it. Obviously (to Dan), the instructors who led the workshop knew about it before he ever had a chance to encounter the new, meaningful interpretation. Moreover, once he formed this deep understanding he could ‘see it everywhere’ (textbooks, Base Ten blocks organization, worked-out long-division algorithm examples). All these experiences supported a sensible conclusion: the mathematical knowledge (e.g., algorithm for efficiently dividing numbers of any magnitude) is independent of the knower (e.g., Dan, see Steffe, 1990). More often than not the teacher may not be aware of this epistemological stance or of its implication that anyone, hence one’s students, can perceive (‘see’) the mathematics the teacher came to perceive.

Typically, teachers whose practices seem to be grounded in a perception-based perspective appreciate the difficulties involved in coming to ‘see’ abstract mathematical concepts. However, for these teachers the process of learning is essentially not problematized. Rather, those teachers conceive of learning as a straightforward transition from not ‘seeing’ to ‘seeing’ the mathematics the teacher now ‘sees.’ Fostering such a transition becomes the teacher’s goal; a task is a tool for accomplishing that goal—clearly and most efficiently pointing to and revealing the piece of mathematics to students. This is a key reason why such teachers embrace reform-oriented, child-centered methods, which lead to a classroom ecology that differs markedly from traditional classrooms in terms of *how* learning is fostered. Yet, the knower-independent epistemological stance common to both traditional and perception-base perspective entails analogous response to the teacher’s ongoing question, “*What* should I teach next?”

Consider a teacher who has robust understandings of the mathematical terrain to be learned by students. She or he may also understand developmental landmarks that researchers found to underlie student progress (e.g., Dan knew that quotitive division is conceptually less advanced than partitive division, whereas partitive division was more compatible with the long-division algorithm). The teacher assesses that a group/class of students is (a) yet to understand or (b) already understands concept “X.” In the former case the teacher intuitively teaches concept “X” because students do not ‘see’ it. In the latter case the teacher moves to fostering students’ ‘seeing’ of the next-in-sequence “Y” concept. The intuition for doing so is sensible if one assumes that students already see concept “X” and do not yet see “Y.” In short, within traditional or perception-based perspective pedagogies one intuitively focuses not on showing students what they already ‘see’ but rather on teaching (revealing) what they don’t.

Two interrelated reasons seem to be at the root of this intuitive tendency (see von Glasersfeld, 1995). First, it is rooted in a deep presumption about human communication, where people customarily assume that the sense others make of what they utter is compatible with one’s own sense. Second, when people form mathematical conceptions that underlie their ‘seeing’ of the world in a certain way they most often cannot ‘return’ to ‘seeing’ it without those conceptions. Coupled with the knower-independent epistemological stance they naturally attribute to fellow humans the unproblematic capacity for the same ‘seeing.’ Consequently, in spite of the seemingly different methods, instructional tasks are used in both traditional and reform-oriented as a means for showing students the mathematics in equivalent way to how the teacher ‘sees’ it.

Traditional and perception-based perspectives were distinguished from an epistemologically different approach termed conception-based perspective, which draws on Piaget’s (1985) key notion of assimilation and the implied, learning-problematizing notion of the *learning paradox* (LP, Pascual-Leone, 1976). If assimilation is determined by a person’s extant conceptions, how can anyone ever form more advanced conceptions? In particular, how can students assimilate tasks/activities in which a teacher engages them to promote learning of a new (to them) mathematical idea unless they somehow have already established conceptions that afford this assimilation?

Teaching rooted in a conception-based perspective draws on Piaget’s explanation of how reflective processes (specifically, reflective abstraction) enable construction of new mathematical ideas as transformation (accommodation) in learners’ assimilatory conceptions (Steffe and Wiegel, 1992). This explanation underlies an examination of an epistemological stance needed for successfully teaching mathematics (and mathematics teachers) that I recently introduced, termed *Profound Awareness of the Learning Paradox* (*PALP*) (Tzur, in press). In-depth discussion of *PALP* goes beyond the scope of this paper. However, it suffices to stress that teaching rooted in *PALP* begins with engaging students in tasks and activities that encourage them to independently use mathematics they already know (i.e., concept “X”). To foster learning of the intended piece of mathematics (concept “Y”), a teacher uses tasks as a means to (a) let students use their available conceptions for setting a goal and initiating an activity to accomplish this goal, (b) orient their attention to effects of the activity that differ from what students anticipated, and (c) relate the newly noticed effects with the activity in an anticipatory way (see Simon and Tzur, 2004; Tzur and Simon, 2004). That is, a task enables student construction of a new regularity (conceptual invariant) as transformation in their previously available conceptions (Steffe, 2002). This approach entails not only that tasks do not have agency (Watson, this forum), but also that tasks do not directly and straightforwardly reveal the new idea to students; rather, tasks indirectly occasion their mental constructive processes (Mason, 1998; Pirie and Kieren, 1992).

From an epistemological standpoint, then, I distinguish two cases of how interpretations of MTs and MTEs may differ regarding task enactment. In the first case, a MT’s traditional perspective is incompatible with the MTE’s perception-based perspective, that is, both are unaware of the learning paradox. When interpreting MTs’ task management the MTE is likely to focus on shifts in student and teacher (inter)activity. In the second case, the MTE’s conception-based (*PALP*-rooted) perspective is incompatible with either a MT’s (2a) traditional or (2b) perception-based perspective. When interpreting MTs’ task management the MTE is likely to focus on both the nature of student activities and the limitations of teacher attempts to straightforwardly engender student ‘seeing’ of the intended mathematics. Thus, the MTE analyzes how a teacher’s plan and implementation of tasks reflect a host of teacher anticipations regarding how students’ work on the task might (or might not) bring forth their learning. Most importantly, the MTE can apply the *PALP* to the teachers’ work and potential growth, that is, consider how teachers’ anticipations structure (afford and constrain) their assimilation and interpretation of events that demonstrate the extent to which a task enabled students’ progress (Tzur, 2007). Thus, the MTE’s analysis will regularly focus on how (a) the interplay between anticipated and actual teaching-learning events and (b) the teacher’s regulation of her or his anticipation—explain task modification.

Implications

The analysis presented above, regarding *one of the important reasons* for task transformation (teacher epistemological stance), has three important implications. First, researchers who study teacher development can greatly benefit from being cognisant of their own epistemological stance relative to teachers’ stance (e.g., case #1, 2a, or 2b). This provides researchers with a tool for inferring into teachers’ assimilatory conceptions of how/why they use insightful tasks (see Krainer, 1999). For example, when working with Dan I was able to not only avoid denouncing his desperate shift to traditional methods, but also to figure out what could be a continual assimilatory barrier to his sense making of my co-teaching and co-planning with him. Articulating a teacher’s epistemological stance, when coupled with the MTE’s application of *PALP* to *teacher learning*, informs the design of mathematics teacher education tasks that are likely to promote teacher progress from a perception-based to a conception-based perspective. For example, I found Dan’s questioning to be rooted in conceptions that could be transformed into novel separation between his own mathematical models and his models of student thinking, which thus became my goal for his learning. This implication is relevant to Herbst’s (this forum) emphasis on teachers as stakeholders accountable for the task being instrumental to the institutional goals of teaching. I argue that a teacher’s sense of accountability necessarily includes an implicit or explicit view of (a) what constitutes learning and (b) why a particular way of managing a task, in the specific here-and-now of an unfolding mathematics lesson, is likely to foster it.

Second, this analysis can assist researchers in identifying the necessary minimum shift in teachers’ epistemological stance so that task modifications are not detrimental to the quality of student mathematics. Key here is the postulation of a continuum along which teacher epistemological stances may emerge. While certainly desired, fostering teacher progress to the higher end of the continuum can prove very difficult (Tzur et al., 2001). Focusing on conceptually feasible shifts is likely to require articulation of individual teachers’ epistemological stances, but it will assist the MTE in finding a sound starting point for the desired shift. Moreover, it can contribute to a scholarly examination of how might teacher development of intended epistemology be promoted. For example, Watson’s (this forum) proposal to engage teachers in the design of tasks seems to be conducive to teacher shift from the middle to the upper end of the continuum, because of the need to use the task as an explicit link between the intended mathematics and assumed student extant conceptions.

Last but certainly not least, this paper pointed to a critically needed shift in MTEs’ stance. Case #1 above indicates that a MTE may identify a task modification without being aware of epistemological stances. In this sense, my analysis sheds light on a profound awareness that we as a community of MTE need to develop and embrace. To borrow from Steffe’s (1995) distinction between first and second order models of mathematics, I ask: How can we promote MTEs’ progress toward a conception of teacher task management that clearly distinguishes between the MTE’s own (first) order model of task characteristics/pedagogy and the MTE’s second order model of mathematics teachers’ models? In this regard, I agree with Sullivan’s (this forum) suggestion to engage MTs in analysis of non-routine and conventional tasks, particularly because it can foster the MTEs’ reflection on and comparison between what makes specific task pedagogies easier/harder for the MTs (and for students).

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task transformation is the teacher’s responsibility

Anne Watson

University of Oxford, UK

There is a resurgence of interest in task design as an important factor in mathematics teaching. Design has to be taken seriously not only for extended, multi-stage, authentic assessment tasks but also for the very ordinary things we ask students to do day-to-day in classrooms. For example, in lesson study, microanalysis of those aspects of the object of learning emphasised by teachers shows that, even when using very similar tasks, affordances for learning can be quite different.

I distinguish between task and activity, and claim that it is the teacher’s professional task to adapt and select tasks. I compare two lessons to show how task-specific-pedagogy can make a difference to learning, even when there is shared design and agreement about the nature of mathematics and learning. Thus teachers are positioned as designers, and task design needs to be a focus of mathematics teacher education.

What do I mean by task?

While some authors see ‘task’ as referring only to complex, multi-stage, exploratory problems, such as a problem situation (Brousseau 1997, p. 214), designed over time, I include any statements, materials, questions, incidents which are expected to impel certain kinds of activity in the classroom. This includes deliberately designed situations, and also the prompts and questions constructed in lessons by teachers and learners. It is natural then to look at task sequences because the way small tasks are strung together structures activity just as much as the interactive moves associated with longer tasks. I choose this distinction because it allows me to focus on task as a tool (see also Tzur, this forum), and to talk of the mathematical activity that ensues in classrooms as both influenced by and influencing the tasks (see Christiansen and Walther 1986).

A task has no agency. It is a tool alongside other tools, designed for a hypothetical purpose, but which only becomes purposeful when it is used and adapted by a teacher and also by the learners. A task, like any tool, on its own does not have purpose, except latently in its design. It becomes purposeful in activity through human agency. Thus the task becomes transformed by classroom activity and can also transform classroom activity through affording particular kinds of engagement. It is possible through principled design to make different kinds of activity possible, yet it is also possible to imbue a casual task (such as a question made up on-the-fly) with rich mathematical purpose through teaching. Purpose is as much a feature of pedagogy as it is of task, so analysis of what happens has to conjoin task and pedagogy. The relationship between task and task-specific-pedagogy is most informative about how the relationship between task, teaching and learning is seen and enacted.

This activity-theoretical analysis explains why purposes of activities with tasks are transformed by teachers, and also that this must be so. Cases where no transformation takes place can now be seen to be special, and may even demonstrate insensitive teaching, silencing learners’ voices. But this analysis does not show the nature of these transformations, nor even how tasks can manifest goals, only that tasks afford certain kinds of activity.

Herbst (this forum) distinguishes between task as representation of mathematics and the task as opportunity for students to learn. This is a helpful distinction because it can be used to question the assumed hegemony of designers’ intentions. A designer may know more mathematics, or have a more research-informed understanding of how task and learning might relate but, as Herbst shows, there is a complex management task to be done including attention to institutional factors such as examinations, timetables, and broader factors such as establishing norms. Herbst therefore champions descriptive rather than normative theories of teaching.

Teaching Geometrical loci

Several teachers in the same school were teaching groups of 12 year-olds who were more or less similar in previous attainment. All teachers agreed to use a similar approach to teaching loci using a combination of straight-edge-and-compass construction tasks and the physical whole class activity of ‘acting out’ loci, They would ask students to work on paper to ‘find all the points which satisfy a given rule’ and to follow physical instructions to ‘find a place to stand so that ….’ (e.g. ‘find a place to stand so that you are the same distance from these two points’). Teachers agreed that all classes would construct, both physically and with conventional tools, circles, perpendicular bisectors of line segments, and angle bisectors and some other loci. Teachers also agreed not to use the word ‘locus’ until students had a sense of what it meant as the set of points that ‘follow’ or ‘satisfy’ the given rules. They also agreed that ‘angle-bisector’ should be introduced as ‘points which are the same distance from these two lines’ so that angle-bisection was seen as a result. An indoor open space was available to do the physical task, and teachers used this at different points during their lessons. The teachers discussed and agreed the overall aim that students would relate their physical experience of standing according to rules to the processes of geometrical construction. This relationship is partially obscured by the affordances of the physical task: it is possible to take a ‘gap-filling’ role without constructing a personal interpretation of the instructions, and hence not to have an experience of being a point in relation to other points to refer to when reproducing the locus on paper. It is also worth mentioning that these students had little experience of geometry beyond some knowledge of angles, and the naming of polygons – no classical, formal, geometry at all.

Five lessons taught by five teachers as a result of this co-planning process were filmed. The mediational devices (words, artefacts, actions, images) and instructions used by the teacher and other students, whether intentional or not, shape the learners’ experience of the lesson. In interactive lessons such as these, the mediational tools of language and purposeful tool use are also shaped by the learners. For example, while a ruler affords measuring and line-drawing activity, learners’ take-up of these affordances is different in different tasks. We even saw three students using two rulers to create an ad hoc, set square to ‘test’ whether a particular angle might be 90 degrees.

I shall describe similarities and differences between two lessons to show how task-specific-pedagogy contributes to learners’ different experiences. Both teachers used a mixture of asking, prompting, telling, showing, referring students to other students’ work and so on. They focused on getting students to explain their choices and actions. All students had to work out as much as they could themselves about how to do the constructions, by reasoning and by listening to each other’s reasons. The tasks were presented in remarkably similar ways offering similar variation in similar ways in terms of language, gestures, and statements of aims. Teachers’ intentions, as reported to each other and to us before the lessons, were similar. Written work was similar, a combination of rough sketches and accurate diagrams; all teachers praised accurate constructions. In one class they also had to write statements describing similarities between the tasks. One could loosely describe these lessons as being models of good modern mathematics teaching practice, with respect to both classical mathematical validity and current ideas about social norms to enable learning.

Analysis of variation, situational norms, questions and prompts, and the demands on learners provided very similar results. Yet as a mathematical observer I knew that the mathematical affordances of the lessons varied; they provided different kinds of intellectual and mathematical engagement. Teachers offered the components of the tasks in different orders; teachers emphasised different things to students at different times; there was a range of different patterns of participation for individual students in each lesson; there were different kinds of tool use. I do not have space to describe all of these but will focus on critical differences.

***Lesson A***

In this lesson, the physical activity took place first, with teacher *A* emphasising ‘same distance’ throughout the various loci. Some students observed the action from a balcony to have an overview of the final shapes achieved. Students then returned to the classroom and were asked to construct the same loci as had been acted out physically. Throughout her small-group interactions the teacher repeatedly used the phrase ‘same distance’.The physical activity happened first so that students were expected to use their memory of the physical actions when they came to make constructions in pencil and paper. No public instructions for constructing were given; instead students were asked to work out how to do them. The teacher worked with small groups of students asking them what they remembered and how they could reproduce it. In general she said ‘you can use the compasses’ when students needed to join points, sometimes showing them how to do it and then asking them to do it again for themselves. The emphasis was on collections of points, each of which has a particular property to do with ‘same distance’, and on joining them using the compasses.

***Lesson B***

The task sequence started with students working out, as a class, how to use a pair of compasses to construct circles, perpendicular bisectors and angle bisectors. Teacher *B* repeatedly referred to compasses as the tool for reproducing equal lengths: he said this himself, and also asked students ‘what can we use to get equal lengths?’ and ‘what do compasses do for us?’ and ‘why would I use the compasses?’ He invited students to demonstrate their ideas on the board, and also used the strategy of placing ‘wrong’ points to encourage students to understand the role of constraints. The teacher reinforced the power of the tool by comparing its role in constructing the two different bisectors, so that students were looking at the positions of, and relationships between, equal lengths in the constructions. By taking this approach, learners were able to talk about relationships within the diagrams as if they were caused by the equal lengths, rather than equal lengths merely being a drawing method. It was possible for them, by this focus, to get a sense of classical geometry. Then the physical activity took place with all students. After that they had to produce statements about the connection between the physical and construction activities.

Comparing Differences

In lesson *A* the emphasis had been on sets of points and ‘same distance’, in *B* the emphasis was on constrained trajectories and the comparison of activities. I interviewed the teachers a year after these lessons, having triggered their memory with videos. Each teacher was still teaching the same class. In case *A* they had recently returned to the concept of loci. Students remembered the lesson and some time during the intervening year had connected the physical and constructing experiences for themselves. The teacher believed this to be due to her use of similar language throughout to link tasks. She talked about how hard it is for students to understand the two-way implications of loci, that all points following a rule can be spatially represented (in these cases by lines) and that any point on these lines therefore followed the rule. Rather than being explicit about this she had chosen to emphasise ‘same distance’ in each context. Teacher *B*’s overriding memory of the lesson was the difficulty students had in reproducing individually the constructions they had developed as a group, compared to the strong qualities of their statements about the relationship between the tasks.

From a mathematical content viewpoint both lessons were successful in promoting significant and lasting learning about loci. Each invited learners to shift from obvious, intuitive visual and physical responses to the more formal responses required for mathematics. In each of these lessons there were emphases on relationships between variables, properties, reasoning about properties and relationships among properties, so hierarchies based on assumptions about complexity do not identify difference.

***Task/activity differences***

In these lessons, interpretations of the task have been made by individual teachers, after team planning. There is no reduction of challenge, in the terms offered, such as that reported by Stein and her colleagues about adoption of research-informed tasks (Stein et al., 1996).

What differed was what was emphasised by the teacher, but I am not saying that this was merely talk. Rather, the difference was, I claim, due to the underlying general relationships within which the teacher saw the task as being embedded. Because teachers see these differently they therefore use different language, different sequencing and different emphases so that different comparisons and connections can be made. This sense of different, but equally valid, mathematical activity around the same concepts does not, for me, appear to be captured totally in Tzur’s reasons for different task adaptation. Tools were used differently, but we do not know if this was due to deliberate choice or not. Views of how students come-to-know mathematics were similar. The institutional and management issues are similar. But in mathematical terms Teacher *A* talked about a two-way relationship between points and lines, and how this is also an issue with graphs as representations of functions. Meeting this duality with loci would make it easier to recognise the duality with other graphical representations. Teacher *B* saw the comparison between tasks as being an example of looking for similar structures in disparate experiences. These two groups of students would therefore be differently prepared for future mathematical activity.

***Working with teachers***

Sullivan (this forum) uses ‘non-routine’ tasks with teachers to think about sequencing, prompts to enable and extend mathematical activity, explicitness about desirable aspects of activity, and the development of a community in which it is habitual to compare and reflect on methods and results as new objects of study.

So far I avoided the assumption that the teacher is somehow deficient in relation to the designer-researcher, but in Sullivan’s paper the focus is explicitly on how teacher educators can work with teachers on incorporating explicit kinds of designed tasks into their teaching. The description ‘non-routine’ has well-understood implications for task-type, yet what we have found in the UK is that all task-types can become routinised by reducing engagement to a sequence of instructions for action. Pre-service teachers often have embedded assumptions of what it means to ‘do’ ‘non-routine’ tasks because in school they followed limited rubrics for assessment purposes. I would extend Sullivan’s definition to include tasks ‘not readily solved by the application of a familiar process’.

Watson and Mason (2007) list aspects of task-use in teacher-education settings that are common throughout the world. This includes asking teachers to: work on mathematics; then use similar tasks in practice (extended, comparative, multi-stage, realistic, or exercise tasks); analyse task structures and observe lessons using them; observe, analyse, compare teaching and learners’ work using similar tasks. From these practices, it is the comparisons that are most likely to expose different conceptualisations of mathematical ideas because, as shown above, variation shows up best against a background of similar practices (Watson and Mason 2005, Tzur this forum). The role of the task in these educational practices combines Herbst’s distinctions between task as representation of the mathematics and task as tool for fostering learning. In all these cases, and in Sullivan’s, the task itself is given.

Another way to engage teachers with tasks is to involve them in the design process, an approach taken by Swan (2006) in his study of improving mathematics teaching. Swan’s study encompasses all stages of the design process, from theoretical and experimental design, through systematic trials of tasks and pedagogy with teachers, and re-design. He then designed training for teachers to use the tasks, and researched the effects of using the tasks in teaching. He identified changes in ‘typical’ teaching while using these ‘new’ tasks and found that the nature of change depended strongly on the teacher’s previous practice. While nearly all teachers changed from less to more learner-centred approaches, their practice and their students’ experiences still differed significantly. A major difference in the final teaching was whether teachers were able to move from a desire to control students’ engagement with content, to giving free rein to processes of ‘conceptualisation’. Teachers were using the same task in the similar ways, but ultimately what appeared to make the most difference was whether the teacher believed that learning happens by making sense of confused and conflicting experiences. This difference influenced whether they used such tasks a lot or a little, whether they adapted the task types to other topics, and how they managed the ‘closure’ of such tasks. Keitel (2006) sees explicitness of purpose and value as a crucial ingredient in teachers’ use of tasks, and Sullivan (this forum) sees explicitness about pedagogy as also important. For example, in Watson and Sullivan (in press) we point to the importance of discussion after a task has been completed to enable learners to relate their experience to their developing mathematical repertoire, and to the conventional canon. To do this convincingly a teacher has to believe that a task as she sets it affords learning of an appropriate kind, even where tasks are the product of rigorous design research, as Swan’s were.

Teacher as designer

Anecdotally, we hear people saying that ‘teachers get in the way’ and that the aim is to create ‘teacher-proof materials’, yet the above examples question whether seeing task designers as custodians of mathematical meaning makes any sense in practice.

Institutional and cultural components of teachers’ decision-making have to influence task adaptation (Herbst, this forum); prevailing classroom norms also make a difference to learners’ experiences, even when teachers are trying to change. These factors alone guarantee that teachers will adapt and design tasks for their own purposes, and Tzur (this forum) draws attention to change in such adaptation due to epistemological differences. I have added further differences, namely teachers’ conceptualisations, how they see mathematical ideas embedded in relation to other ideas, and how these emerge in task-specific pedagogy.

A non-teacher-designer has to ensure that tasks afford the most possible intellectual challenge, such as reasoning about properties and relationships, or adaptations of skills and techniques used in unfamiliar contexts. However, as the authors in this forum show, the teacher is not a neutral conduit for tasks but is also a designer.

It makes sense, therefore, to work with teachers on task design rather than only on task implementation. Karp (2007) reports how task design was a major aspect of teacher preparation in his work in the former Soviet Union. Prestage and Perks (2007) do this with pre-service teachers, turning the issue on its head, showing how teachers can identify the limitations of designed tasks and use them to develop richer teaching through task-specific-pedagogy.

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