Commentary on  
Elastic Multiplication

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### Elastic Multiplication

Although it is often said that ‘multiplication is repeated addition’, in fact ‘repeated addition’ is (only) one form of (discrete) multiplication. Multiplication also includes ‘scaling’. Indeed, repeated addition is a discrete form of scaling. If something is scaled by one factor, and then that is scaled by a second factor, the overall effect is a scaling by the product of the scale factors. Marking points on an elastic, and then stretching it, preserves the relative positions of the marks, and so instantiates scaling. Comparing where marks move to in relation to their original position on the un-stretched elastic leads to operations with fractions, or indeed scale factors generally.

This task has been evolving over a considerable period of time. Anne Watson had seen Ulla Runnesson using elastic to display fractions as actions on objects. Trying to develop tasks to exploit this led to extending the elastic to ask further questions. Several attempts to work on geometric scaling based on what the French call Thales’ theorem (similar triangles have proportional sides) along the lines of Frog-Scaling (Dennis & Addington 2011) failed to satisfy me or open the way to substantive challenges for students. The result is an applet and its associated tasks which are still in the process of development. A similar approach was developed independently by Roger Harvey (2011).

The applet was developed because the use of physical elastics is quite demanding and difficult to handle in plenary, even with a helper. Individuals need to make marks on an elastic, and make a ‘standard’ version on paper or use a rule against which to compare. Furthermore, common elastics don’t actually stretch all that much, and arranging for a specific point to be invariant is particularly difficult.

#### Narrative

It takes people a while to enter into the world that is offered. Watching the stretching and contracting of the upper elastic provides an action (dragging) with an effect (what changes; what now aligns) to make sense of. Natural tools for this include ‘same and different’, or what is much the same thing, looking for what is invariant and what is permitted to change and over what range. The initial mode is expounding, as both the development of the applet and the initiation of the task match this mode in the same way that any preparation for or initiation of a session does: using the virtual, imminent or actual presence of students to bring the teacher into special contact with the mathematics. The applet manifests this relationship.

Some of the initial time is taken up in gazing at the whole, discerning details, and establishing relationships among those details so as to develop a narrative which gives an account-of what happens when things are dragged. The shift to developing a narrative to account-for what is perceived is sense-making (literally and figuratively).

As sense-making begins, the applet-manifested phenomenon mediates between the student and the mathematical relationships in a mode of exploring. Once the expository mode has been displaced by an exploratory mode, students are taking the initiative either to seek underlying relationships or to work empirically. Even where the applet is used in plenary, a period of individual or group ‘work’ can provide an experience of a shift of mode. The applet acts as more than a result-producing phenomenon like a function machine, because there is an underlying geometry being resonated concerning scaling. It is manifesting structural relationships.

A shift into exercising arises when people want to check their generalisation or ‘theory’ against other specific examples. When links are made, relationships recognised, the mode can shift again into expressing, as the applet’s phenomenon initiates a desire to express those structural relationships. For some students, and for most under the guidance of a teacher, a shift can be made to perceiving properties (generalisations) which are being instantiated in the various ‘examples’. But this requires students to act as if, perhaps even to realise, that there are multiple instances of structural relationships (varying the fixed point is a further dimension of possible variation).

Once people have a reasonable narrative for what is happening, specific questions can be asked about predicting where a specified point (*R*-image in the screenshot) will align after a specified scaling, or its reverse, predicting the scaling when the alignment of the *R*-image is known. If used in a workshop setting, people can then spend time with pencil, paper and perhaps elastic to try to work out not only the given situation, but how to work it out for any configuration of scaling, *R*-source and *R*-image alignment. The applet provides multiple examples, but the most effective route is to use the examples to identify the underlying relationship, rather than to work empirically. If people do work empirically in order to locate a generalisation, then there is the opportunity to press them to justify their conjecture in terms of the elastic and its stretching (rather than in terms of generalising a table of values).

It would be unwise to expect that a period of time working on or with the applet would lead to facility with fractions, or even a rich understanding or appreciation, but it could supplement and enrich other situations in which fractions and their arithmetic have been encountered. The onus then would be on the teacher to prompt withdrawal from the action at various times in and various situations, in order to make sense of that action and to look for connections with other experiences of fraction arithmetic. The aim is to promote student education of their awareness, in the sense of facilitating the coming-to-mind of a suitable action in a future situation.

#### Affordances

In addition to experiencing fractions as operators on objects, the applet brings to the fore multiplication and division of fractions by fractions, in terms of doing and undoing scalings. Varying the fixed point incorporates addition and subtraction of fractions as well. Again, doing and undoing, also known as conjugation, is a helpful way to reason about the effects of scalings. As with any micro-world, only certain actions are possible, so these constraints focus attention on those actions. Experiencing fractions as actions (so that rational numbers are the effect of fraction-actions on a unit) could help students overcome the standard confusions about fractions (Floyd *et al*. 1982) which are often treated as if they were numbers, when it is equivalence classes of fractions that correspond to numbers.

#### Extensions

The applet begins with one the left end fixed, but there is an option to choose where the fixed point of the scaling is on the original elastic. The initial specific question is intended to initiate a collection of actions as students interact with the applet to try to get-a-sense-of the underlying structure, making use of their own experience of elastics, but cast in the perhaps less familiar formal form of scaling. A reasonable goal is for students to engage in the activity of developing and articulating a general statement of where on the original a given point goes under a given scaling with a given fixed point, and what scaling is employed to send a given point on the original to some other given point on the original using a given fixed point. The applet together with the experience students can call upon, perhaps brought freshly to mind by using an elastic themselves, provides suitable resources and tasks, self-generated in detail but constrained within the micro-world of the applet. Any comprehensive response involves adding and subtracting as well as multiplying and/or dividing fractions.

There is a topological result implicit in the activity, namely that as long as the stretched elastic overlaps both ends of the original elastic, there must be an invariant point, a point which does not move during the stretching.