**Functions represented as linear sequential data: relationships between presentation and student responses**

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*This study investigates students’ ways of attending to linear sequential data in two tasks, and conjectures possible relationships between those ways and elements of the task design. Drawing on the substantial literature about such situations, we focus for this paper on linear rate of change, and on covariation and correspondence approaches to linear data. Data sources included a survey instrument of six tasks that was developed in collaboration with a group of teachers, and the tasks for this paper are two concerned with linear functions. The whole survey was given to 20 students from each of UK years 7–11 and 10 students from each year 12–13 (total of 120 students). Our analytical approach was to identify what all students appear to do, not how correct they were or what pre-determined methods they might use. Our analysis uses theories of dual-process and dynamic graded continuum to suggest conjectures about how students' capabilities in acting with sequential data depend to some extent on task features, as well as on curriculum and pedagogy.*

Keywords: covariation; correspondence; rate of change**;** generalising linear functions; sequences; task design

**1. Introduction**

The function concept is both an explicit and implicit foundation for advanced study in mathematics itself and as a tool, and the roots of function understanding do not consist of a single hierarchical pathway (Schwindgendorf, Hawks and Beineke 1992). Several decades of intensive research on functions have yielded much information about functions learning and teaching, including accumulating research that reveals substantial difficulties, and ways of addressing them (e.g., Dreyfus and Eisenberg 1983; Even 1998; Herbert and Pierce 2012; Leinhardt, Zaslavsky and Stein 1990; Tall and Vinner 1981; Thompson 1994; Yerushalmy 1991, 1997). The research tells us a lot about difficulties, but learning more about students’ particular ways of seeing functions in different stages of school could also contribute to curriculum and pedagogic choices.

Our overall project begins to address this issue, aiming to construct a description of progression towards functions throughout the secondary mathematics years based on probing students’ understanding of key contributory concepts. In this paper we investigate students’ responses to two sequence tasks involving linear rate of change, and covariation and correspondence approaches to linear data. We conjecture how students' capabilities in acting with sequential data can be influenced by presentation and representation of the task and the questions posed. An underlying assumption is that learning depends on multiple factors, among them are the written curriculum, school and classroom context, teaching, and possibly on the level to which teachers are ‘functions aware’ (Watson and Harel 2013) and national assessment regimes. The curriculum in the UK has an informal approach to functions, not requiring a formal treatment until year 12 for those who continue to advanced study, but younger students will, for example, generalise sequential data and meet input-output models as ‘function machines’. We developed a survey instrument over several design cycles working closely with experienced teachers to adapt and develop tasks that matched curriculum aims and the development of functions concepts (Swan 1980; Wilmot et al. 2011). As well as working on the instrument and anticipating student responses, teachers also conducted the survey, and provided insights about pedagogy to inform our discussion. These teachers are ‘functions aware’ due to their first degree level of mathematical knowledge. This survey was implemented by the teachers in two schools in the UK (ages 11 to 18) to provide data for analysis about the development of function concepts, while being aware of grouping, teaching, curriculum, prior attainment, and other variables. Most of the students involved were those they taught themselves.

We shall outline our research approach and demonstrate its application in the two tasks involving linear functions. Both present situations that provide sequential data. One task focuses on rate of change. In the second task students have to provide an algebraic generalisation for a growing sequence of spatial structures – such tasks are common in UK textbooks and assessment tasks and are specifically mentioned in the national curriculum. Our design and analysis cycles led us to relate the capabilities and tendencies demonstrated to elements of task design.

**2. Theoretical Background**

The theoretical background consists of three parts. It begins with a review of relevant research dealing with functional relationships, covariation and correspondence in particular, and the concept of rate of change. Then it focuses on findings from research related to sequential tasks. The last part discusses some theoretical perspectives on responses to tasks. Our use of literature developed in an integrated and cyclic fashion with our task design and analysis processes. In particular, our thinking about dual-process theory (e.g., Kahneman and Frederick 2002) and the dynamic graded continuum (Cleeremans and Jiménez 2002) is a response to a need to explain certain features of students' task responses and relate them to the task presentation.

**2.1 Functional relationships: Covariation, correspondence, and rate of change**

Two general approaches to creating and conceptualizing functional relationships are often discussed in the literature: a correspondence approach and a covariation approach (e.g., Confrey and Smith 1994, 1995; Slavit 1997). These approaches are not distinct, but rather “... shed different light on the common underlying functional relationship.” (Dorfler 2008, p.147). The correspondence approach builds a rule for determining the unique *y*-value from any given *x*-value, thus builds ­correspondence between *x* and *y*. A correspondence relation, such as an input-output model, is a plausible contribution towards understanding that relations between two sets of numbers can sometimes be expressed as general algebraic ‘rules’ and our expectation is that students using this approach talk about how operations on *x* generate *y* values. A covariation approach to functions involves an understanding of the manner in which the dependent and independent variables change. It entails being able to coordinate movement from *ym* to *ym*+l with movement from *xm* to *xm*+l. With tables of values, it involves the coordination of the variation in two or more columns as one moves down (or up) the table. A covariation approach involves analysing, manipulating, and comprehending the relationships between changing quantities. Covariation is a precursor to calculus and also fits well with modeling natural phenomena, where data typically consists of changes in a phenomenon. Indeed Blanton et al. (2011) characterise functional thinking as “generalising relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior” (p. 47). Although we do not agree that this is an all-encompassing characterisation, since there are many functions for which covariation does not make sense (e.g., Dirichlet Function, a function which assigns a child to her/his mother), it is a helpful perspective on functional ideas typically met in school mathematics. We expect a student using this approach to mention changes in variables.

A closely-related aspect to covariation understanding of functions is rate of change (Carlson et al. 2002; Confrey and Smith 1994, 1995; Thompson 1994), and in some literature these ideas are used interchangeably. We distinguish between them in this way: covariation is the awareness that changes in two or more variables can be related to each other within a natural phenomenon, or within a mathematical situation; rate of change is an instantiation of such a relationship in which changes in one variable can be expressed formally or numerically in terms of changes in another variable. Rate of change could therefore be correctly deduced through a procedural approach in stereotypical situations, such as applying a formula or reading a graph gradient, without any awareness of what it means in terms of covariation, for example being unable to deal with varying rates of change (Herbert and Pierce 2012); alternatively a student might express a situation in terms of covariation but be unable to operationalise this idea as a rate of change.

The covariation approach is considered to be harder to understand than correspondence because the focus is on change rather than quantity (Mevarech and Kramarsky 1997). Likewise, rate of change is known to be a complicated concept for students (Herbert and Pierce 2012). Bell and Onslow (1987), for example, found that secondary students had considerable confusion concerning the numerator/denominator roles of the two quantities comprising a rate. Further, relationships between covarying quantities can be reasoned in various ways without grasping the idea of dependency (Confrey and Smith 1995). Research has shown that children possess intuitive ideas about functional relationships, such as dependence, causality, and variation, which have been developed through observations of physical phenomena that surround them every day (e.g., Confrey and Smith 1994). Children notice that certain things "go together" in the real world and expect that things will change over time and that there will be some pattern to that change (e.g., the change of temperature over time). Thus, there is an argument for not delaying the idea until students are older. The UK curriculum at the time of the study stated that students from year 6 upwards should learn to "use linear expressions to describe the nth term of an arithmetic sequence, justifying its form by referring to the activity or context from which it was generated" (Department for Education 1999). The associated guidelines distinguished between a 'term-to-term' rule and a 'position-to-term' rule in its elaborations. However, in school tasks students tend to find the differences in the dependent variable, and sequence tasks tend to mask the need to pay attention to the independent variable (Confrey and Smith 1994).

**2.2 Sequences tasks**

Our survey starts with tasks involving linear sequential data for several reasons: Finding and using patterns is an important strategy for mathematical problem solving; younger students would be familiar with linear sequence tasks and hence there would be access for all; pencil-and-paper graph plotting techniques and spreadsheet techniques tend to involve sequential data; data arising from phenomena are often presented in sequential form. Dealing with sequential data therefore has pedagogic importance towards understanding functions but is also an end in itself.

There is a considerable body of literature about sequential tasks, (e.g., Carraher, Martinez and Schliemann 2008; Dörﬂer 1991; Radford 2006, 2008; Stacey 1989). Studies of processes of generalising sequences vary in the types of patterns, the population studied and their perspectives and accordingly in the categories of generalisations they present. For example, Rivera and Becker (2008) distinguish between constructive and deconstructive forms of generalisation when the sequences are spatial. The constructive form results from perceiving ﬁgures as consisting of accumulations of non-overlapping parts, exhibiting the standard linear form *y* = *mx* + *b*. The deconstructive form is based on initially seeing overlapping sub-conﬁgurations in the structure and would lead to separately counting each sub-conﬁguration and taking away parts (sides or vertices) that overlap. In their study, deconstructive reasoning was found to be more difﬁcult.

Most studies agree that reaching theoretical generalisations is difficult and students tend to begin with a term-to-term approach to the dependent variable. After ﬁnding the constant difference and answering any concrete questions correctly they may attempt to generalise by continuing to apply additive strategies. Stacey (1989) defined four main generalisation approaches in linear contexts: counting from a drawing; finding a common difference between pairs of consecutive terms and using it to add from term to term; making a proportional assumption; finding a statement that involves both multiplication and addition. She noted that many students tend to construct rules with simplicity rather than accuracy in mind, for example, by improperly applying direct proportional reasoning. Students’ attraction to simple rules was evidenced in the significant percentage of students who swapped from a correct linear model to an incorrect direct proportion model for harder parts of a question. Such inconsistency led her to conclude that “students grab at relationships and do not subject them to any critical thinking” (Stacey 1989, p.163).

Several researchers have suggested that task presentation influences students’ approaches since presenting data in order might encourage a term-to-term approach (e.g., Orton, Orton and Roper 1999; Stacey 1989; Steele 2008). They noticed that when students are presented with spatial or numeric sequences, usually accompanied by a table of values to fill, the pervasive strategy is to calculate the difference between pairs of terms to ﬁnd the successive differences, failing to develop general rules. The move from this to covariation is not trivial (Blanton and Kaput 2011; Smith 2008).

The aim of the studies above was to focus on obstacles to the generalisation process. It cannot be claimed, however, that students who succeed have a sense of functions, even though some authors describe this as a ‘functional approach’. To have a sense of ‘function’ would require variation of functions and comparisons between them and among their properties (Carraher et al. 2008). We are not setting out to make claims about students' understanding of functions from their performance on our two sequential tasks. Rather we are interested in what we can learn about capabilities that make some small contribution to a later understanding of functions.

**2.3 Theoretical perspectives on responses to tasks**

We found some inconsistencies in students' responses: individuals may show the requisite domain-specific knowledge and skills in some parts of a task but not in others; individuals may appear to 'grab at relationships' in one place but reflect and apply critical thinking in another. Responses to any tasks are not a straightforward window into students' capabilities. Psychological approaches to learning offer various theories to account for students' apparent failure to apply knowledge; two main theoretical approaches to this phenomenon in mathematics education are those of Fischbein and Vinner. Fischbein’s (1987) theory of intuitions emphasized that some errors may be the consequence of pervasive, self-evident and immediate intuitions that interfere with reasoning. Fischbein distinguished between primary intuitions which ‘develop in individuals independently of any systematic instruction as an effect of their personal experience’ (p. 64) and secondary intuitions which ‘are acquired, not through natural experience, but through some educational interventions’ (p.71). Vinner (1997) developed an alternative theoretical framework to account for what he calls ‘meaningless behaviors’ in mathematical contexts. He pointed at pseudo-analytic processes in which students superficially select elements in the problem and apply a procedure relevant for a typical question due to superficial similarity with previous problems. Vinner suggested that these pseudo-processes are ‘simpler, easier, and shorter than the true conceptual processes’ (p. 101). Based on the minimal effort principle (i.e., the preferable procedure to achieve a goal is the one with the least effort) many students therefore (unconsciously) ‘prefer’ these processes.

To explain different responses as described by Vinner (1997), some researchers in mathematics education have offered dual-process theory (DPT) (Leron and Hazzan 2006, 2009). They draw on cognitive psychology to account for erroneous approaches when students do have the necessary knowledge and skills (e.g., Epstein 1994; Evans and Over 1996; Kahneman and Frederick 2002; Stanovich 1999). DPT distinguishes between immediate reactions and more reflective, thoughtful, responses that draw on knowledge. The theory proposes that behaviour and cognition operate in parallel in two quite different modes, called System 1 (S1) and System 2 (S2) respectively. S1 processes are characterized by immediacy, high accessibility, automaticity and effortlessness. S2 processes are slow, conscious and effortful. In school mathematics this split involves the secondary intuitions that are the result of education (Fischbein 1987). In most day-to-day situations S1 does background work, creating space for S2 to operate. In school mathematics, some styles of teaching, learning and assessment can favour S1 – the generation of quick habitual responses based on secondary intuitions that may be incorrect - and S2 may or may not operate. Because secondary intuitions the result of education, they can be associated with past experiences of success in different contexts, or with different teachers, or simpler versions of a task, or with immature concept images based on limited experience (Tall and Vinner 1981).

An approach that avoids duality is the dynamic grading continuum (DGC) suggested by Cleeremans and Jiménez (2002), which focuses on the representations used in tasks. Different kinds of response to tasks can then be explained in terms of how much conscious control a student can exercise on the representations, or whether the representation itself exerts a strong influence on the student's actions. It is unhelpful to assume that students doing tests or surveys are always exerting full conscious control over their response to particular representations, because past experience will have led them to take aspects of representations for granted, and visual appearance can trigger the kind of automatic responses which are not dissimilar from the shortcuts developed by learners in familiar symbolic contexts. This visual triggering is more complicated and unpredictable than pure responses to perceptual stimuli, because the triggers are symbolic and therefore require some decoding[[1]](#footnote-1). For example, Gillard et al. (2009 a&b) show that inappropriate proportional reasoning on arithmetic word problems can result from S1 processing rather than from shortcomings in knowledge and skills and this is relevant for our study. Ainley and Pratt (2005) observe that "almost all proportional tasks students encounter at school are formulated in a missing-value format ... students tend to develop a strong association between this problem format on the one hand and proportionality as a mathematical model on the other hand." (p. 98). The DGC framework suggests that differences in representation generate variation in forms of reasoning without assuming a multiple-system framework. The framework also distinguishes between implicit and automatic reasoning; by contrast, dual-process theorists use the terms interchangeably (for discussion, see Osman, 2004).

The responses that appear to be controlled by the representation are comparable to S1 systems – cheap on mental energy, memory and processing effort. DGC theory suggests that the choice of representations and the control they afford are important, and this accords closely with Duval's (2006) view that "Mathematical comprehension begins when coordination of registers starts up" (Duval 2006, p.126) because to coordinate registers requires some control over their use.

DPT and DGC do not only apply to erroneous reasoning because they also explain the automatisation that takes place for skilled mathematicians (Krutetskii 1976). We find the S1/S2 distinction helpful in thinking about these phenomena and shall use them throughout this paper (while recognising that other frameworks are available).

At the task design stage our anticipations were informed by: pedagogic experience, knowledge of typical errors, and analysis of components of function understanding. However, at the analysis stage we found that a combination of DPT and DGC theories applies to design as well. Our problem as task designers and analysts of students' work is that the operation of S1 is effortless while S2 is effortful, so students may not make the extra effort unless something in the situation alerts them to such need. To find out as much as possible of students' capabilities, tasks have to scaffold S2 responses by making them *necessary* beyond S1 responses. We attempt this through use of representations, presentations and questions that may inhibit automatic response and arouse students’ conscious control. Representation can be in verbal, numerical, diagrammatic, graphical or symbolic form, while visual, positional and juxtapositional presentation might also affect response. For example, a situation can be represented numerically while presented in sequential or non-sequential order. A question can be posed in a representation that relates directly to a situation, thus triggering an automatic response, or in another representation, thus requiring some conscious action. Tasks presented in familiar representations, with questions in the same representation, might lead to S1 responses in which students act according to the representation, and S2 responses might arise as a result of being unable to act directly with the representation, or being given an unfamiliar representation. Balancing these issues with accessibility was our aim.

**3. Methods**

**3.1 Population and sample**

To meet the aims of our overall project we wanted data from a wide age-range of students who are most likely to need to understand functions in later study or employment. We needed a sample that was large enough to encompass a wide range of possible responses, while being small enough to analyse individual responses in detail and take the full range of responses into account. The survey was given to two suitable classes from each of year 7 to year 11[[2]](#footnote-2) – a highest achieving class and a middle achieving class – and also to first and the second years of post-16, when studying mathematics is optional. The classes were from two schools, each school providing data from alternate years. The teachers gave the survey to all students in each class and then selected random anonymised samples of 10 scripts from each class, chosen from an alphabetic list proportionally. In this way we received 20 scripts from each UK year 7 to 11 inclusive, and 10 scripts from the first and second years of post-16 mathematical study (total of 120 scripts). The schools and teachers were similar in many ways: size 1000±200; socio economic factors, i.e. the proportion of children ever having free school meals was about 17%; ethnicity, i.e. the proportion having English as an additional language was about 2.5%; stability and qualifications, i.e. all staff were qualified to teach mathematics and most had been at the school 3+ years. Both schools were non-selective state-maintained serving a mix of urban and rural populations.

**3.2 Task design**

First we developed a concept map of functions seen from both pure mathematical and modeling perspectives. We elaborated this map using the curricula of the countries we were going to use for the full study (Department for Education (1999) for the UK), and the literature as outlined above. It was from this conceptual investigation, which cannot be reported for reasons of space, that rate of change as one key developmental focus appeared in all possible pathways of development towards functions understanding, being associated formally with gradient and early calculus and informally with shapes of graphs. Other key ideas included graphing and algebraic representation, not relevant for this paper. For linear sequential situations, we wanted to know if students could use rate of change rather than 'constant difference', and whether they constructed general rules by following patterns or in other ways. To achieve this we disturbed the usual term-to-term S1 responses, as documented in research, by “breaking” the difference in the *x* values (Task 1) and providing data in a non-sequential order for Task 2. We reasoned that this presentation would encourage students to take the independent variable into account and hence encourage covariational approaches; in Task 1 we drew attention to the disturbance by asking for a relevant value for the dependent variable. The questions had to be accessible for students in years 7 to 13, so with the teachers we chose familiar types of tasks, but varied the representations so that students might not respond habitually to them. The language used was agreed with the teachers, but the UK teachers did not think younger students would have a mathematical understanding of 'rate'.

Through trials we scaffolded students' reasoning to encourage them to use the formal understanding of rate of change in Task 1 and achieve full generalisation in Task 2 (English and Warren 1998; Stacey 1989). Such scaffolding might have expanded capabilities as well as revealing existing capabilities, but as our purpose was to describe capabilities in action, as possibilities, this was not a problem. Task 1 (see Figure 1) asks students to read a table that represents the number of seconds it takes for a lift to descend. Question 1.1 requires finding a numerical relation between a given number of seconds and the corresponding floor number according to a given numerical pattern. The ‘break’ in the lift journey was designed to help students notice that the term-to-term vertical pattern of the table requires thought, and the missing dependent value was to draw attention to the break so that they would not be controlled by the downward linear pattern. Question 1.2 asks explicitly for rate, which is not defined; choices are given so that some comparison can be made with the given data, and/or the data table needs to be transformed into 'floors per second'. We expected this to encourage S2 responses, especially as the dependent variable is not given in unit steps. The last part of the task is a comment saying “You might want to use it to check your answer to 1.1” in order to 'nudge' critical reflection.

Task 2 (see Figure 2) asks students to interpret a geometric pattern that represents the perimeter of a chain of hexagons. It was adapted from Wilmot et al. (2011) and is a typical spatial sequence situation. Question 2.1 requires finding numerical relations between given numbers of hexagons and the corresponding perimeters according to a geometric pattern. There is no table of values and data are not given in numerical order – this avoids a presentation which influences term-to-term S1 responses. Question 2.2 asks for a general computation method to obtain the perimeter of a large number of hexagons. This question cannot be answered directly from the data; a relationship has to be inferred which cannot be found without transforming given data and comparing it to the diagram. Question 2.3 asks for an algebraic expression to describe the perimeter of any number of hexagons. This is an explicit request for a new representation and no direct support is given, but the previous questions might direct students towards understanding the relations and structure they need to express. Question 2.4 asks for an explanation, a transformation into words. In most of these questions we expected students to have to coordinate the representations and generate S2 responses. Task 1 focuses on rate of change first implicitly and then explicitly. Task 2 asks for a general expression which could be developed in a number of ways, such as using correspondence or covariation.

Teachers anticipated that students of all ages would spot the correct missing value in 1.1, but they expected students to have some problems with 1.2. 'Rate of change' as a taught idea first appears in the curriculum in year 12. They expected students to be fairly competent at finding formulae in task 2 since such tasks were familiar in school. In general, teachers' comments appeared to be about finding linear formulae through understanding step size in the dependent variable and then using input-output numerical reasoning using step size as the multiplier. This approach would match what they taught.

**Task 1**

You are staying in a hotel on its 14th floor. You are going to use the lift to go down to the parking level. The hotel has a ground level numbered zero, and there are several parking levels underneath the zero floor.

The table below shows what floor you reach after a number of seconds.

|  |  |
| --- | --- |
| Number of seconds | Floor number |
| 0 | 14 |
| 2 | 10 |
| 4 | 6 |
| 6 | 2 |
| 7 | ? |

**1.1** Where will the lift be after seven seconds?

1. 1
2. -2
3. 0
4. -1

Explain your answer.

**1.2** At what rate does the lift descend?

1. 0.5 floors per second
2. Four floors per second
3. Two floors per second
4. One floor per second

Explain your answer.

**1.3** You might want to check whether your answer to question 1.2 fits with your answer to question 1.1.

**Fig. 1** The Lift Pattern task

**Task 2**

For the following geometric pattern, there is a chain of regular hexagons (meaning all 6 sides are equal):



**2.1**

For **1** hexagon the perimeter is **6**.

For **3** hexagons the perimeter is **14**.

For **2** hexagons the perimeter is \_\_\_\_\_\_\_\_\_\_\_\_\_

For **5** hexagons the perimeter is \_\_\_\_\_\_\_\_\_\_\_\_\_

**Note: perimeter is the number of outside edges.**

**2.2** Describe the process for determining the perimeter for 100 hexagons, without knowing the perimeter for 99 hexagons.

**2.3** Write a formula to describe the perimeter for any number of hexagons in the chain (it does not need to be simplified).

**2.4** Explain why you think your formula in question 2.3 is correct.

**Fig. 2** The Hexagon Pattern task

**3.3 Data Analysis**

We looked for identification of rate of change in task 1 and methods of constructing a generalisation for spatial sequential data in task 2. However, we were more interested in the different approaches that students had taken, whether correct or not, to give us insight into their understanding of functions at this level. We read all the sample scripts individually, noting the approaches used and extracting the students' verbal explanations for each task. We developed categorisations of these approaches as evidenced in their answers and written explanations, constantly checking these against the whole data set and testing distinctions to see if all responses fitted into one, and only one category. This process required several passes through the whole data. The analytical process was thus iterative and comparative. Although we first took an open approach to possible approaches, it became clear that categories associated with covariation and correspondence were adequate for our purposes, so we looked again for evidence of covariational and correspondence perspectives and also at treatments of rates of change, and tested our interpretations further against the data. We also looked for patterns of behaviour between different subtasks. We analysed responses separately for each school year, but found very few differences between the approaches taken by students, or in their use of rate of change, across the age range or across past levels of achievement, although there were small differences in success as students grew older and also between average and higher achieving groups. We present results from the whole sample together, and supplement this with comments about any relevant age-related differences where they occur.

*Task 1: The Lift Pattern task*

We coded each student’s responses liberally according to ideas related to functions from the literature and team experiences as mathematicians and teachers. We classified these responses according to broad approaches to functional reasoning evidenced across the whole sample. This process led to three categories:

1. Term-to-term approach
2. Term-to-term followed by covariation approach
3. Covariation approach

Table 1 presents the categories, accompanied with examples of student responses.

A response was categorized as a *term-to-term approach* if it focused on the differences in the dependent variable (i.e., number of floors), without mentioning the changes of the independent variable (i.e., number of seconds). Example 1 in Table 1 illustrates this approach.

A response was categorized as a *term-to-term followed by covariation approach* if it expressed a *term-to-term* *approach* when addressing the question of finding the missing value in the table, and then, when were asked to explicate the rate, moved to *covariation approach*, i.e., explicitly mentioned to the ways in which the two quantities change in relation to each other. Example 2 in Table 1 illustrates this approach.

A response was classified as *covariation* *approach* if it explicitly mentioned the ways in which the two quantities change in relation to each other in both questions. Example 3 in Table 1 illustrates this approach.

Table 1: Categories related to reasoning about rate of change

|  |  |  |  |
| --- | --- | --- | --- |
| Category # | Category description | Example # | Example  |
| 1 | Term-to-term approach | 1 | 1.1 [-2] because from 14 and downstairs the floors are going down in 4’s.1.2 [Four floors per second] because the elevator graph shows that it skips 4 levels. |
| 2 | Term-to-term followed by covariation approach | 2 | 1.1 It will be -2 because it is going down in fours so you just had to take away four from 2.1.2 I think its two floors per second because if you go to floor 12 it will be 1 second. |
| 3 | Covariation approach | 3 | 1.1 The lift will be at 0 after 7 seconds because its rate is 2 floors per second. It generally goes 2, 4, 6 but 7 is one second so zero is counted as a whole floor!1.2 The rate is two floors per second because the floor number descends by 4 : 2 seconds, so for every 2 floors : 1 second. |

*Task 2: The Hexagon Pattern task*

Two separate iterative and comparative analysis processes were implemented, one to describe approaches to functional reasoning and another to classify success in achieving a correct generalisation.

In the first process we looked for evidence of all attempts to make relationships between data items, since in such tasks these relations could tell us something about students' understanding of functions (remembering that 100 out of the 120 have not been taught anything formal about the function concept). We coded each student’s responses according approach, whether used correctly or not. We then classified these responses according to evidence across the whole sample, in the manner described above. Table 2 presents the categories, accompanied with examples of student responses.

A response was coded as *no conceptualization of functional relationship* if it did not involve relating two quantities, e.g., counting with no reference to number of hexagons. Example 1 in Table 2 illustrates this approach.

A response was classified as expressing a *correspondence approach* if it described a relation (one-to-one in this case) between the two sets of numbers (i.e., the number of hexagons and the perimeter). Examples 2 and 3 in Table 2 illustrate such approach as they describe the relation between the number of hexagons and the perimeter in terms of multiplication and addition. The response given as example 2 suggests constructive generalisation, using components of the whole shape in terms of one variable, whereas the response given as example 3 suggests deconstructive generalisation, seeing the shape as made of elements that overlap (Rivera and Becker 2008). These examples mention structure to specify the correspondence. However, there were other correspondence responses without reference to structure, merely applying wrong proportional reasoning, e.g., example 4 in Table 2.

In distinction from a *correspondence approach* response which focuses on the relation between two quantities, a response was classified as a *covariation approach* if it focused on the relation between the changes in two quantities (i.e., the change in the number of hexagons and the change in the perimeter). Such responses included expressions such as "for each one more… it goes up by…", "for every time you add… you add". Example 5 in Table 2 illustrates this approach as it explicitly coordinates two varying quantities. While this example takes starting points into account, there were other covariation responses without paying attention to starting points, as illustrated by example 7 in Table 2. Examples 5 and 7 used constructive reasoning, taking 4 as the step size, and there were other deconstructive responses using a step size of 5, subtracting one for the overlaps, as illustrated in example 6.

Finally, a response was classified as *correspondence followed by a covariation approach* if it expressed a *correspondence approach* when addressing the question of finding the perimeter of 100 hexagons, and then, when were asked to generate the formula for any number of hexagons, moved to *covariation approach*. Examples 8 and 9 in Table 2 illustrate this approach.

Table 2: Categories related to approaches to functional reasoning

|  |  |  |  |
| --- | --- | --- | --- |
| Category # | Category description | Example # | Example  |
| 1 | No answer or “I don’t know”. |  |  |
| 2 | No conceptualization of functional relationship: Empirical methods involving counting. | 1 | 2.2 You count how many edges in the chain.2.3 & 2.4 No response |
| 3 | *Correspondence approach*  | 2 | 2.2 Because the edges join up they all link together but the 2 end hexagons have 5 sides so you do 5\*2 then the hexagons in the middle have 4 sides so you do 98\*4 then add the two together.2.3 & 2.4 2\*5, then how many hexagons in the middle of the end ones \* 4. Because if you had 3 hexagons together the end ones have five sides 5\*2 then the middle one has 4 so 1\*4 then add the answers together = 14. |
|  |  | 3 | 2.2 You would have to multiply 6 (the number of sides) by 100 (number of hexagons) = 600. Because some sides are joined you have to take them away.2.3 & 2.4 6\*(number of hexagons) = [space] – number of joint sides. Because if you multiply the number of sides by number of hexagons and then subtract the number of joint sides it will be correct. |
|  |  | 4 | 2.2 You time a perimeter of 1 hexagon by 100. 2.3 & 2.4 Hexagon = 6 \* number of hexagons. Because you time the perimeter of 1 hexagon by the number of hexagons. |
| 4 | *Covariation approach*  | 5 | 2.2 You are adding 4 every time you add a hexagon so you would multiply 4 by 100 then minus 4 and add 6 for the first hexagon.2.3& 2.4 ((4*n*) - 4) + 6.  |
|  |  | 6 | 2.2 When you add 1 more hexagon, you add 5 sides, but lose 1, so you add 4 each time. The perimeter of 1 hexagon is 6, so you need to add 2.2.3 & 2.4 Perimeter = 4*n* + 2 as described in 2.2. |
|  |  | 7 | 2.2 It goes up in 4’s for each one hexagon.2.3 & 2.4 *a*\*4 = perimeter. All of the hexagons perimeters show a pattern, they all go up in 4’s for each one more hexagon. |
| 5 | *Correspondence approach* *followed by a* *covariation approach*  | 8 | 2.2 To find the perimeter of 100 hexagons you can multiply the perimeter for 1 hexagon by 100. This works as it's in proportion. Example: 1 hexagon – perimeter of 6. 100 hexagons = perimeter of 600.2.3 & 2.4 *p*(*n*)=(*n*)-1. For every one shape added, one side of the hexagon is lost. |
|  |  | 9 | * 1. Get one hexagon and times it by a hundred.

2.3 & 2.4 *p*(*n*) = *n*\*4. Because the perimeter increases by 4 for every 1 hexagon so it is number of hexagons multiplied by 4. |

The other focus for analysis was on generalisation, taking correctness into account. Each student’s response was categorized according to its generalisations: (1) no correct generalisation of any kind; (2) generalisation expressed correctly in verbal terms only, or (3) generalisation expressed correctly verbally as well as algebraically. The responses in the examples 1, 3, 4, 7, 8, and 9 in Table 2 were coded as category (1). The response in example 2 was coded as category (2). The response in examples 5 and 6 in Table 2 were coded as category (3).

**4. Results**

*Task 1: The Lift Pattern task*

Table 3 presents the distribution of the categories related to reasoning about rate of change.

Table 3: Distribution of the categories related to reasoning about rate of change

|  |  |  |
| --- | --- | --- |
| **Category # and description** | **# responses** | **%** |
| 1. Term-to-term approach | 20 | (17%) |
| 2. Term-to-term followed by a covariation approach | 33 | (28%) |
| 3. Covariation approach | 67 | (55%) |
| **Total** | **120** | **(100%)** |

As shown in Table 3, 100 out of 120 students were able to choose and explain the rate of change and interpret the formal terminology of 'rate' and 'floors per second' for question 1.2. These ideas appeared formally in the UK national curriculum only for the older 20 students, so capability does not derive from formal teaching. Interestingly, many of these responses were of category 2: In these cases, students chose the wrong answer of -2 when asked to find where the lift will be after seven seconds, explaining that the lift is going down 4 every second. However, when asked to find the rate at which the lift descends, they used both variables to get the correct answer (see example 2 in Table 1). Despite the stimulus in question 1.3, there was no evidence of any rethinking. A much smaller number of responses showed a term-to-term approach throughout the whole task (code 1) (see example 1 in Table 1). Use of term-to-term declined with age.

*Task 2: The Hexagon Pattern task*

The second and the third columns of Table 4 present the approaches to reasoning in students’ responses. The triples in the fourth column show the distribution between generalisation types (# no correct generalisation, # generalisation expressed correctly verbally only, # generalisation expressed correctly verbally and algebraically). The fifth column lists the corresponding proportions.

Table 4: Distribution of approaches to functional reasoning

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Category # and description** | **# Responses** | **%** | **# Generalisation types** | **%** |
| 1. No answer | 11  | (9%) | (11,0,0) | (100%, 0, 0) |
| 2. No conceptualization of functional relationship  | 4 | (3%) | (4,0,0) | (100%, 0, 0) |
| 3. Correspondence approach | 62 | (52%) | (37,3,22) | (60%, 5%, 35%) |
| 4. Covariation approach | 26 | (22%) | (11,0,15) | (42%, 0, 58%) |
| 5. Correspondence approach followed by covariation approach | 17 | (14%) | (17,0,0) | (100%, 0, 0) |
| **Total** | **120** | **(100%)** | **(80,3,37)** | **(67%, 2%, 31%)** |

As shown in the second and the third columns of Table 4, the most common approach (52%) to conceptualizing the functional relationships was *correspondence* with students suggesting a general rule for the relation between the number of hexagons and the perimeter. 40% of these did so correctly, some only verbally. The *covariation approach* of coordinating changes in the quantities was less popular (22%), but 58% of these were successful. Responses expressing a *correspondence approach* when addressing the question of finding the perimeter of 100 hexagons, and then *covariation approach* when generating a formula for any number of hexagons, constituted 14% of the responses, but none of these achieved a successful generalisation.

The strongest connection between method and success in building generalisation was with the *covariation approach.* Those who did not succeed using a *covariation approach* failed because they did not take starting values into account, as can be seen in example 7 in Table 2 in the data analysis section. Those who did not succeed with the *correspondence approach* either assumed proportionality or took a deconstructive approach but were vague about the subtraction required, as can be seen in Table 2, examples 4 and 3 correspondingly. Those who took the *correspondence followed by covariation approach* began with assuming proportionality, and then moved to *covariation approach* without taking starting values into account (see examples 8 and 9 in Table 2). There could be some successful 'covariation then correspondence' approaches that we were unable to discern, because the multiplier in a successful correspondence approach is also the unit step size of a covariation approach, and this approach would have matched what teachers said about their experience. Our analysis depended on students' written explanations, so unless this was stated we could not take it into account. Note that only the older 16% of students would have been taught formally about functions.

**5. Discussion**

In our discussion we are going to conjecture relationships between the results, processing, and features of the design. For Task 2 it appeared that most students started with a correspondence approach, unlike research results that show a term-to-term approach to be most frequent (e.g., Stacey 1989). This is likely to be due to the data not being offered in sequential form, so that term-to-term responses to sequential data tables were not triggered by visual appearance. For many students the assumption of proportionality, which we assume is a schooled S1 response, was enacted when asked for perimeter for 100 hexagons, and there was nothing in that part of the question to urge S2 systems into action. From the teachers we know that students are familiar with generalising spatial sequences and there is classroom discussion of the need to look at the structure or the starting values rather than merely multiply, but, as Fischbein (1987) pointed out, faulty intuitions can persist and the proportionality assumption does so in particular. Although we had not presented this as a missing number problem, many appeared to treat it as such, as Ainley and Pratt suggested (2005). Indeed our teachers confirmed that in their teaching on proportionality they emphasise layout of data as a tool to draw attention to relationships. We conjecture therefore that question 2.2 was treated as a missing number problem because that was an option offered by the layout of the data, triggering a schooled S1 response.

14% of students gave faulty proportional reasoning for question 2.2 but then showed that they *could* analyse structure – although not taking starting values into account and not correcting their earlier flawed reasoning – when prompted to construct a formula. Their change in approach provides evidence that the S1 response of assuming proportionality could give way when question 2.3 required a complete change of representation.

The most successful route proportionally was covariation, comparing increases in perimeter to increases in numbers of hexagons. Both constructive and deconstructive reasoning was used to do this, using either 4 or 5 as the step size. Teaching methods can confuse covariation with term-to-term reasoning (Blanton and Kaput 2011), leading to failure with a term-to-term approach, and Task 2 does not tease this out because the independent variable is the natural numbers, and possibly because we could not discern some covariational reasoning. However, the fact that some students used 5 as the step size suggests that not all were using term-to-term reasoning alone. Our presentation made it necessary for students to use S2 systems to think about the sequential relationship rather than assume it from a data table, so they were more likely to coordinate visual and numerical representations to get information, including consideration of outcome values, and less likely to rely solely on gaps between adjacent numbers. This would explain the higher levels of success among students who chose this approach. Students seemed to have made the shift from term-to-term reasoning to understanding covariation – not a trivial step (Blanton and Kaput 2011; Smith 2008). Our evidence from Task 1 also supports this view.

We now return to Task 1. In this task students often began with term-to-term reasoning, possibly not looking at the independent variable. Thus, the ‘break’ in the lift journey, so that the term-to-term vertical pattern of the table needs thought, seemed not to disturb all S1 responses to familiar number patterns (although use of term-to-term declined with age, as Stacey (1989) also found). However, most students developed a full covariation approach when specifically asked about rate, a question which could not be directly related to the representation, but needed comparison of data. Although term-to-term reasoning would appear to be connected to covariational reasoning because each requires searching for differences, it is obviously not the same thing in this task yet these students whose answers to the first part could be construed to mean a lack of knowledge nevertheless were able to handle rate without unit change in the independent variable when asked to do so. In this task, rate is not something that can be understood by building up, as it can in task 2. Instead, comparing step size in the dependent variable to non-unit step size in the independent variable is a significant contribution to later understanding of gradient (Orton 1983).

Teachers, and some authors, claim that rate and covariational reasoning are difficult for students, yet many students did use rate, or a rate-like comparison, in tasks 1 and 2. Students' success at finding rate in task 1, even when their teachers did not indicate the importance of covariation, questions research that says this is difficult, and also questions the teachers' expectations, yet conforms to the view of Confrey and Smith (1994, 1995) that young students can have an understanding of covariation and rate.

Overall, the findings suggest that approaches to sequential data appeared to be dependent on presentation and representation rather than solely on either S1 or S2 tendencies. Approaches were also dependent on the question requirements and how closely that question matched the presentational forms of the data. This dependency included considerations of familiarity, secondary intuitions, and whether the questions make S2 reasoning necessary. Design of tasks to probe students' understanding therefore has to disturb S1 responses, and, while we had attempted to do that in tasks 1 and 2, we suggest that there was still enough similarity with past experiences of similar tasks that the disturbances we had introduced were not necessarily noticed at first and many first responses were of the type we had hoped to avoid. Comparison between tasks 1 and 2 suggests that if students are familiar with the independent variable being the natural numbers they are unlikely to pay attention to it unless urged to do so. However, when asked questions about the same data that required transformation of representations, students could overcome S1 and use S2 systems. We notice that S2 appeared to be triggered when we changed representation or required coordination, and hence control, of representations in our questions.

When we presented these findings to teachers their comments took a different perspective, not of students' capabilities but of how their responses related to teaching. Their explanations for relative success in Task 1 were not based on students' capabilities to deal correctly with rate, but: 'students have met rate in science', 'may be familiar with movement of lifts', or 'could follow the 'floors per second' prompt'. All teachers said they did not explicitly teach rate, but that there were topics in which language associated with rate (e.g. 'so much per so much') was used. They did not, for example, see performance an indication that dealing correctly with rate and covariation in a non-sequential linear context might provide a foundation for an earlier focus on rate in their teaching. All the teachers taught linear spatial sequences by relating the step size of the dependent variable to the multiplier in a correspondence approach (often called position-to-term), and while this might involve some covariational language this could be redundant as the independent variable is always natural number. This could explain both the more popular correspondence approach from those who did not realise the importance of step size, and the relative success of those who used covariation.

**6. Conclusion**

There are three main findings, bearing in mind that both tasks were linear:

(1) Results from task 1 showed that many students were able to handle rate in a situation in which the step size was not unitary, before their curriculum and teaching included this idea.

(2) Results related to Task 2 challenge existing research that a term-to-term approach is the most common response. In our study, a correspondence approach was the most common, but those who took a covariation approach, which involves coordinating a term-to-term approach to the dependent variable with the step-size of the independent variable, and taking a starting point into account, were the most successful proportionally. The task presentation avoids superficial term-to-term responses and enabled us to see more deeply what students were capable of doing with such data.

(3) In both tasks, a considerable number of students varied their approaches according to the questions, usually starting with an S1 approach triggered by appearance or familiarity and changing when the question did not match the representation given, requiring an S2 approach. Stacey (1989) reports changes of approach but in her data students switch to incorrect proportional reasoning (possibly S1) from a previous correct linear model. These switches are therefore appearing to depend on presentation of the situation and the questions. These results deserve further examination in future research.

We do not claim that our results contradict other studies. Instead we point to task design that requires students to go beyond S1 responses and thus reveals more about their capabilities in relation to sequential data. In particular, we have found evidence that our students were more capable with rate and covariation approaches in linear contexts than has been suggested elsewhere, and indeed than was suggested by their teachers at the design stage. There are implications for the design of early linear function experiences if students are to have the best opportunity to develop their functions understanding.

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1. We refer readers to the work of Radford (e.g., 2006, 2008) for a full semiotic analysis of some similar tasks. [↑](#footnote-ref-1)
2. year 7 in UK are aged 11-12. [↑](#footnote-ref-2)