**Adolescent learning and secondary mathematics**

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Abstract:

In this paper I develop my thinking about how learning secondary mathematics can relate to the adolescent project of negotiating adulthood. All too often it does not, yet the same kinds of adolescent autonomous thinking which so often lead to disaffection and rejection, not only of mathematics but of school and life more generally, can be embedded and enhanced positively within the teaching and learning of mathematics. I suggest parallels, and some kinds of tasks which enhance the adolescent project through mathematics, and mathematics through adolescence.

.**Introduction**

*To confirm the deepest thing in our students is the educator’s special privilege. It demands that we see in the failures of adolescence and its confusions, the possibility of something untangled, clear, directed (Windle 1988)*

My personal rationale for taking this view lies in my work as head of a mathematics department during the late 80s and early 90s in a school which served a socio-economically disadvantaged area. In this school, academic achievement was generally low. Nevertheless, our mathematics results were usually the best of all subjects, sometimes ahead and sometimes just behind those achieved in creative arts, and nearly 100% of the cohort, including persistent absentees, achieved some kind of mathematics grade. I take little pride in this, however, because the teaching methods we used, while appealing to the students and to us at the time, did not enable students to do as well as those at nearby middle class schools. While students were learning *some* mathematics, we nurtured and depended on their powers of exploration, application, and their natural enquiry to achieve success. We were praised for this in some quarters, because we had developed ways of working on mathematics *in* school which were similar to the ways in which quantitative problems arise *out* of school. However, we did not enable the majority of students to contact the essentially abstract, structural, understandings which characterise the subject in its entirety. We did little to help them engage out of school with new ways of thinking. In Vygotsky’s terms, we failed to support them in engaging with the scientific concepts which can only be deliberately taught (Kozulin 1986, xxxiii). Since then my work has focused on how students with low achieving backgrounds can engage with mathematical thinking, without limiting them to merely being better at workaday thinking in mathematical contexts (Watson, 2006).

**Adolescent concerns**

There is widespread agreement that adolescents are broadly concerned with the development of identity, belonging, being heard, being in charge, being supported, feeling powerful, understanding the world, and being able to argue in ways which make adults listen (Coleman and Hendry, 1990). Naming these as adolescent concerns assumes that Western psychological understandings of adolescence can be taken to be universal, which may not be the case due to the interaction of physiological and cultural factors, but I adopt this understanding for the purposes of this paper. Adolescents engage with these concerns through interaction with peers. This was regarded by Vygotsky as the ‘leading activity’ of adolescence (Elkonin, 1972)), that is the activity which leads the course of development but is not necessarily the only influential activity. Nevertheless, as he saw it, peer interaction is the context in which adolescents work out their relationships with others, adults, the world and themselves. They do this by employing their new-to-them ability to engage in formal-logical thinking, so that they become capable of self-analysis, and analysis of other situations, as internalisations of social consciousness developed with peers (Karpov, 2003).

This new-to-them facility is attributed by Piaget to a biological development of the cortex, yet different societies appear to engender different kinds of world view, different forms of mature argument, and different kinds of abstraction. Vygotsky recognises the role of physiological maturity, and emphasises that the use of the maturing brain is influenced in how the notion of self is worked out by relations in the social world (1986). Thus the biological ability to understand things in more complex, abstract, ways is not the most important influence on learning. Rather, it is how adolescents use adult behaviour and interaction as mediators of their main activity with peers that influences the development of identity and self-knowledge.

School classrooms are thus very important places for adolescents, because it is there that adult behaviour dominates, while the majority present are adolescents whose behaviour and interactions are constrained by adult goals and expectations. It is an easy step from this analysis to ask for more discussion, more work in groups, and more attention to students’ ideas and their social world to produce more engaged learners.

‘Cognitive bullying’ is not too strong a term to describe the kind of teaching which ignores or negates the way a student thinks, imposes mental behaviour which feels unnatural and uncomfortable, undermines students’ thoughtful efforts to make sense, causes stress, and is repeated over time, possibly with the backing of the system or institution. To be forced to revisit sites of earlier failure by, for example, doing fractions during grades 6,7,8 and 9, can, with this perspective, be seen as cognitive bullying which is at best marginally productive and at worst emotionally damaging. Students can become trapped into repeated failure with no way out except to adopt negative behaviour or to accept such treatment compliantly. However, compliance does not imply mental comfort or lack of anxiety. As well as repetitious failure, much mathematics teaching involves showing students how to manipulate and adapt abstract ideas – ways of thinking which are often in conflict with their intuitive notions. Even in the most therapeutic of classrooms this requires ways of seeing and paying attention which are contrary to what the student does naturally – against intuition. Thus for some students the mathematics classroom is a site where natural ways of thinking and knowing are constantly overridden by less obvious ideas. Without time to make personal adjustments, many students give up attempts to make personal sense of what they are offered, and instead rely on a disconnected collection of rules and methods. The term ‘cognitive bullying’ can therefore be taken to mean that students’ own ways of thinking are constantly ignored or rejected, the mathematical experiences which generate fear and anxiety are constantly revisited for repetition, and they are expected to conform to methods and meanings which they do not understand.

This state of affairs reaches a climax in adolescence, when examinations become high-stakes, major curriculum topics become less amenable to concrete and diagrammatic representations, full understanding often depends on combining several concepts which, it is assumed, have been learnt earlier, and adolescents are developing a range of serious disruption habits. Those whose thinking never quite matches what the teacher expects, but who never have the space, support and time to explore why, can become disaffected at worst, and at best come to rely on algorithms. While all mathematics students and mathematicians rely on algorithmic knowledge sometimes, learners for whom that is the only option are dependent on the authority of the teacher, textbooks, websites and examiners for affirmation. Since a large part of the adolescent project is the development of autonomous identity, albeit in relation to other groups, something has to break this tension – and that can be a loss of self-esteem, rejection of the subject, or adoption of disruptive behaviour (Coleman & Hendry 1990 pp.70, 155).

**Enquiry methods**

Stoyanova (2007) reports on the evaluation of an extensive mathematics curriculum project in which students used tasks which encouraged enquiry, investigation, problem posing, and other features commonly associated with so-called ‘reform’ and enquiry methods. Results of analyzing test answers from 1600 students showed very few clear connections between aspects of the programme and mathematical achievement during adolescence. Most notably: problem-posing, checking by alternative methods, asking ‘what if..?’ questions, giving explanations, testing conjectures, checking answers for reasonableness, and splitting problems into subproblems were associated with higher achievement in year 10, while using general problem-solving strategies, making conjectures, sharing strategies were not. Use of ‘real life’ contexts was negatively associated with achievement at this level. Other interesting aspects of her results include that teachers’ beliefs about learners’ ability to learn turned out to be very important, a finding repeated elsewhere (e.g. Watson and DeGeest, 2005), and that explicit teaching of strategies was not found to be associated with achievement, either positively or negatively.

Realistic tasks can provide contexts for enquiry and often enquiry methods of teaching and learning are recommended for adolescent learning. Historically, mathematics has been inspired by observable phenomena, and mathematicians develop new ideas by exploring and enquiring into phenomena in mathematics and elsewhere. It is also possible to conjecture relationships from experience with examples, and thus get to know about general behaviour. But mathematics is not *only* an empirical subject at school level; indeed it is not *essentially* empirical. Its strength and power are in its abstractions, its reasoning, and its hypotheses about objects which only exist in the mathematical imagination. Enquiry alone cannot fully justify results and relationships, nor can decisions be validated by enquiry alone. Many secondary school concepts are beyond observable manifestations, and beyond everyday intuition. Indeed, those which cause most difficulty for learners and teachers are those which require rejection of intuitive sense and reconstruction of new ways of acting mathematically.

School mathematics as a human activity has at least two dimensions, that of horizontal and vertical mathematisations (Treffers 1987).

“In horizontal mathematization, the students come up with mathematical tools which can help to organize and solve a problem located in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, like, for instance, finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries”.(van de Heuvel-Panhuizen 2007)

This shift from horizontal to vertical mathematisation has to be structured through careful task design; it does not happen automatically. A Vygotskian view would be that this kind of shift necessarily involves disruption of previous notions, challenges intuitive constructs, and replaces them with new ways of thinking appropriated by learners as tools for new kinds of action in new situations.

Grootenboer and Zevenbergen (2007) work within a social paradigm which emphasizes agency and identity, yet found that they had to go beyond Burton’s analysis of professional mathematical activity to be informative about what actions learners had to make when working mathematically. In their study, learners had to identify patterns, construct generalizations, use examples to test hypotheses, and identify limits to make progress with a task. By naming these actions, they show how important it is that the intellectual demands of mathematics itself should be taken into account when thinking about teaching and learning. Many of the features of the programme evaluated by Stoyanova could be described as offering agency, but it was seen that only some of them led to improved mathematics learning. The effective features all engaged adolescents in exercising power in relation to new mathematical experiences, new forms of mathematical activity, and being asked to use these, express these, and to display authority in doing so. For example, at year 10 level adolescents responded well to being given authority in aspects of mathematical work: checking answers, giving explanations, asking new questions, testing hypotheses, and problem-posing. These actions appeal to the adolescent concerns of being in charge, feeling powerful, understanding the world, and being able to argue in ways which make adults listen. They offer more than belonging by doing what everyone else is doing, or being heard merely through sharing what has already been done.

**Shifts of mathematical action**

Analysis of student errors in mathematics (e.g. Booth 1981; Hart, 1982; Ryan & Williams 2007) suggest that many students get stuck using ‘child methods’, intuitive notions, invented algorithms which depend on left-to-right reading, or misapplication of verbal tricks. When these methods do not produce the right answers to school questions, often at the start of secondary school, these could well be a contributory factor to rejection of the mathematics curriculum.

The contradictions between intuitive, spontaneous, understandings and the scientific concepts of secondary mathematics can be the beginning of the end of mathematical engagement for adolescents. If they cannot understand the subject by *seeing* what it does and how it works, but instead have to believe some higher abstract authority that they do not understand, then the subject holds nothing for them. This analysis has contributed to the belief that mathematics need not be taught to everyone and that many adolescents only need to become functionally numerate (Bramall & White 2000). But this view misses the point. The authority of mathematics does not reside in teachers or textbook writers but in the ways in which minds work with mathematics itself (Freudenthal 1973 p.147; Vergnaud 1997). For this reason, mathematics, like some of the creative arts, can be an arena in which the adolescent mind can have some control, can validate its own thinking, and can appeal to a constructed, personal, authority. But to do so in ways which are fully empowering has to take into account the new intellectual tools which simultaneously enable students to achieve in mathematics, and which develop further through mathematics (Stech, 2007). To understand this further I present key intellectual tools of the secondary curriculum as illustrations of what needs to be appropriated in order to engage with new kinds of mathematical understanding. Unsurprisingly, these turn out to be aspects which cause most difficulty and my argument is that it is likely that many teachers do not pay enough attention to these shifts as inherently difficult. Not only do these shifts represent epistemological obstacles, in Brousseau’s terms, but they are also precisely those changes to new forms of action which constitute the scientific knowledge of mathematics – that which can only be learnt at school. It is inequitable to expect students to bring their everyday forms of reasoning to bear meaningfully on mathematical problems, when everyday forms do not enable these shifts to be made:

* Shift from looking for relationships, such as through pattern seeking, to seeing properties as defined by relationships
* Shift from perceptual, kinaesthetic responses to mathematical objects to conceptual responses; and a related shift from intuitive to deductive reasoning.
* Shift from focusing on procedures to reflecting on the methods and results of procedures.
* Shift from discrete to continuous ways of seeing, defining, reasoning and reporting objects.
* Shifts from additive to multiplicative and exponential understandings of number.
* Shifts from assumptions of linearity to analysing other forms of relationship
* Shifts from enumeration to non-linear measure and appreciating likelihood.
* Shifts from knowing specific aspects of mathematics towards relationships and derivations between concepts.

A shift towards seeing abstract patterns and structures within a complex world is seen by many psychologists, not only those influenced by Piaget but also the Vygotskian school, as typical of adolescent development (Coleman & Hendry 1990 p. 47, Halford 1999). One of the ways in which these two schools of thought differ is that Piaget appears to be saying that this shift happens biogenetically, and new forms of learning follow. In mathematics, however, it is common at all levels of competent study to move fluently between concrete, diagrammatic and abstract approaches and between examples and generalisations as appropriate for the exploration being undertaken. Vygotsky’s approach is that learners are capable of developing abstract ideas but need interaction with expert others to achieve this for themselves. He recognised that adolescence is a particularly appropriate time for conversations and scaffolding towards abstraction to take place, and that the biological and physical changes which occur at this age also relate to making sense of self in society and self in relation to ideas (1986 p.107).

**Shifts of mathematical action compatible with adolescence**

Over three projects, IAMP, CMTP, and MkiTeR[[1]](#footnote-1) I have observed many lessons with engaged adolescents in which a central feature appears to be the introduction and use of new intellectual tools which reflect new-to-them forms of mathematical action. For example, in one of the IAMP classrooms a teacher would sometimes be very explicit about shifts: ‘you have been doing adding for years; we have to change to thinking about multiplication’.

I shall now give some examples of tasks which generate and nurture mathematical identity in adolescence, while staying focused on the secondary mathematics curriculum as the locus of new forms of thinking. The point of showing these tasks is to demonstrate that, while exploratory and realistic learning environments have much to offer adolescents, it is also possible to structure short, tightly-focused, curriculum-led tasks in ways that lead directly to higher levels of engagement and also employ the social and emotional modes of working that are widely desired.

**Learner-generated examples**

Students in a lesson were familiar with multiplying numbers and binomials by a grid method:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| X | 20 | 7 |  | X | z | +3 |
| 50 | 1000 | 350 |  | 2z | 2z2 | 6z |
| 9 | 180 | 63 |  | -1 | -z | -3 |

They had been introduced to numbers of the form *a ±√b*. The teacher then asked them to choose pairs of values for *a* and *b*, and to use the grid method to multiply such numbers to try to get rational answers. Students worked together and began to explore. At the very least they practised multiplying irrationals of this form. Gradually, students chose to limit their explorations to focus on numbers like 2 and 3 and, by doing so, some realised that they did not need explicit numbers but something more structural which would ‘get rid of’ the roots through multiplication. Although during the lesson none found a way to do this, some carried on with their explorations over the next few days in their own time (Watson and Shipman, in press).

Tasks in which students gain technical practice while choosing their own examples, with the purpose of closing in on a particular property, relationship, or class of objects, can be adapted to most mathematical topics, and when the constraints in such tasks are incorporated into the goal, learners have to engage with new mathematical ideas.

**Another and another …..**

The following task-type also starts from learners’ examples:

Ask students to give you examples of something they know fairly well, then keep asking for more and more until they are pushing up against the limits of what they know.

e.g. Give me a number between zero and a half; and another; and another …

Now give me one which is between zero and the smallest number you have given me; and another; … and another….

Each student works on a personally generated patch, or in a place agreed by a pair or group. Teachers ensure there are available tools to aid the generation – in this case some kind of ‘zooming-in’ software, or mental imagery, would help.

This approach recognises learners’ existing knowledge, and where they already draw distinctions; it then offers them opportunity to add more things to their personal example spaces, either because they have to make new examples in response to your prompts, or because they hear each other’s ideas (Watson & Mason 2006). Self esteem comes at first from the number of new examples generated, then from being able to describe them as a generality, and finally from being able to split them into distinct classes. This is not merely about ‘sharing examples’ but about adopting them and making claims about them; taking note of peers’ contributions, mediated by an adult, and enhancing self-knowledge through this process.

**Putting exercise in its place**

If getting procedural answers to exercises in textbooks is the focus of students’ mathematical work (whether that was what the teacher intended or not) then shifts can be made to use this as merely the generation of raw material for future reflection. Many adolescents have their mathematical identities tied up with feeling good when they finish such work quickly, neatly and more or less correctly; others reject such work by delaying starting it, working slowly, losing their books and so on; still more can produce good-looking work which shows little understanding, their self-esteem tied up with form rather than function. Restructuring their expectations is, however, easy to do if new kinds of goal are explicated which expect reflective engagement, rather than finishing, so that new mathematical identities can develop which are more in tune with the self-focus of early adolescence while requiring new forms of action, reflection on results and processes.

Examples of different ways to use exercises are:

* Do as many of these as you need to learn three new things; make up examples to show these three new things
* At the end of this exercise you have to show the person next to you, with an example, what you learnt
* Before you start, predict the hardest and easiest questions and say why; when you finish, see if your prediction was correct; make up harder ones and easier ones.
* When you were doing question N, did you have to think more about: method, negative signs, correct arithmetical facts, or what? Can you make up examples which show that you understand the method without getting tied up with negative signs and arithmetic?

**Rules *versus* tools**

Student-centred approaches often depend on choice of method, and this, of course, celebrates autonomy. However, mathematics is characterised by, among many other things, variation in the efficiency and relevance of methods. For example, ‘putting a zero on the end when multiplying by ten’ and ‘change side-change sign’ are fine so long as you know when to do these – and mathematicians do not abandon these ways of seeing. Rather than it being a *rule* it becomes a *tool* to be used when appropriate. Adolescents often cannot see why they should be forced to abandon methods and behaviour which have served them well in the past (repeated addition for multiplication; guessing and checking ‘missing numbers’; and so on) to adopt algorithms or algebraic manipulations. On the other hand, they often choose autonomously to abandon past behaviour in the service of new goals. One way to work on this is to give a range of inputs and show students that they can decide which of their methods works best in which situation, and why. This leads to identifying methods which work in the greatest range of cases, and the hardest cases. These ‘supermethods’ need to be rehearsed so that they are ready to use when necessary, and have the status of tools, rather than rules; empowering rather than oppressive. One teacher I have observed calls these ‘bits of technology’ to emphasise that appreciating their usefulness may be delayed.

**Abstract mathematics and adolescence**

In this paper, I have advanced the idea that ways can be devised to teach all adolescents the scientific conceptualizations, and methods of enquiry, which characterise hard mathematics. I suggest that these cohere with and enhance many features of adolescent development. Moreover this can be achieved without resorting to cognitive bullying which is counter-productive and alienating, because the epistemological changes of activity embedded in mathematics are similar to the ways in which adolescents learn to negotiate with themselves, authority, and the world. Agency and identity do not have to be denied, but neither does abstract mathematics have to get lost in the cause of relevance and personal investigation.

In all of the above task-types, students create input which affects the direction of the lesson and enhances the direction of their own learning. Classrooms in which these kinds of task are the norm provide recognition and value for the adolescent, a sense of place within a community, and a way to get to new places which can be glimpsed, but can only be experienced with help. To use the ‘zone’ metaphor – these tasks suggest that mathematical development, relevance, experience and conceptual understanding are all proximal zones, and that moves to more complex places can be scaffolded in communities by the way teachers set mathematical tasks.

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